WAVE INTERACTION IN NON-RELATIVISTIC ELECTRON

BEAM-PLASMA SYSTEMS

by

Gerald A. Otteni

Technical Report No. A-53

July, 1967

Supported in part by the National Aeronautics and Space Administration and by the Air Force Cambridge Research Laboratories under Contract #AF19(628)1699.

ABSTRACT

A theoretical model based on the linearization of the fluid equations and Maxwell's equations for wave interaction in a uniform plasma which is interpenetrated by a nonrelativistic electron beam is developed. The effects of electron-neutral and electron-ion collisions and temperatures of both the beam and plasma electrons are included and no quasi-static approximation is made for the electromagnetic field. An external d.c. magnetic field is assumed to act so that a general formulation is developed which is valid in the limit of small d.c. magnetic fields and in the limit as the field becomes very large. Graphs of the computer solutions are given for the propagation constants in a beam-plasma system for the cases of an unbounded system and for the TM wave solutions that may exist in an axisymmetric cylindrical system in which the finite beam interpenetrates an unbounded plasma.

ACKNOWLEDGMENTS

The author is grateful for the assistance offered by Professor Robert E. Collin for suggesting this topic and for providing encouragement and many helpful discussions throughout the course of this work. The author also wishes to thank Dr. B. S. Tanenbaum for discussions of this problem and Brian G. Gordon for providing assistance in developing the computer program. A special thanks is due Mrs. Christine Griffen for typing this manuscript.

This work was supported in part by the National Aeronautics and Space Administration and by the Air Force Cambridge Research Laboratories under Contract #AF19(628)1699.

TABLE OF CONTENTS

		Page
ABSTRACT		ii
ACKNOWLEDGMENTS		iii
TABLE OF CONTENTS		iv-v
Chapter		
1	INTRODUCTION	1
2	THEORETICAL MODEL	6
	2.1 Introduction	6
	2.2 Development of Linearized Equations	7
	2.3 Electron Beam-Plasma System Equations	16
	2.4 Transverse and Longitudinal Field Separation	22
	2.5 Potential Function Equations	24
	2.6 Transverse Field-Potential Relationships	26
3	UNBOUNDED BEAM-PLASMA INTERACTION	29
	3.1 Introduction	29
	3.2 Mathematical Formulation	30
	3.3 Longitudinal Waves	35
	3.4 Transverse Waves	66
4	BOUNDED BEAM-PLASMA INTERACTION	69
	4.1 Introduction	69
	4.2 The Similarity Transformation	70
	4.3 Beam-Plasma Region	73
	4.4 Plasma Region	77

TABLE OF CONTENTS

Chapter		Page
5	FINITE BEAM-INFINITE PLASMA SYSTEM	80
	5.1 Introduction	80
	5.2 Beam-Plasma Region	81
	5.3 Plasma Region	82
	5.4 Boundary Conditions	83
6	CONCLUSIONS	94
APPENDIX		
A	A.C. POWER RELATIONS	97
B	DERIVATION OF THE POTENTIAL FUNCTION EQUATIONS . .	105
C	BOUNDARY CONDITIONS	116
D	UNIQUENESS OF POTENTIALS	124
E	COMPUTER PROGRAM	128
REFERENCES		142

Chapter 1

INTRODUCTION

The earliest work published on wave interaction in a plasma system which is interpenetrated by a charged beam was that of Langmuir in 1925.[1] The theory was developed specifically to explain the generation of high frequency oscillations in a hot cathode discharge.

Although the parallel plate magnetron was developed in 1921 by A. W. Hull[2] and later refined by many scientists in the 1930's and early 1940's to produce the cylindrical multiple cavity magnetron oscillator, the development of microwave amplifiers proceeded rather slowly. A. Arsenjeva-Heil and O. Heil,[3] in Germany, published their findings of the development of a velocity modulated beam tube which was later given the trade name Klystron. Later, in 1939, R. H. Varian and S. F. Varian[4] and W. C. Hahn and G. F. Metcalf[5] published their investigations of the klystron oscillator-amplifier.

With the advent of World War II, the development of the magnetron and other microwave ampliers and oscillators was hastened due to the obvious need in radar systems. The traveling-wave tube was developed initially by R. Kompfner[6] at Oxford University in 1946, and later studied and improved by J. R. Pierce and L. M. Field of Bell Telephone Laboratories[7] as a broadband amplifier. The backward-wave oscillator was later studied by Jones[8] and Heffner.[9]

Another scientific field that received a great deal of interest during and immediately following the war was plasma physics. Although studies of high frequency oscillations involving stationary heavy ions and electrons had first been published as early as 1906 by Lord Rayleigh[10] who obtained an expression for the electron oscillation frequency, little work was done in this area until the 1940's. Impetus was given to the study of plasma physics initially because of the importance of understanding propagation of electromagnetic waves through ionized layers and the use of plasma devices such as T-R switches and high frequency electron tubes.

Pierce,[11] in 1948, investigated the theory of interaction of an electron beam with an ion cloud to explain the spurious oscillations he observed in traveling-wave tubes. Haeff[12] demonstrated that amplification was possible when one electron beam interpenetrates another. Shortly after this, a number of authors investigated the double stream amplifier problem.[13,14,15]

Since this work, there have been a large number of papers published concerning investigation of the beam-plasma interaction including investigation of stability of the systems, temperatures and collisions in the model and the effects of finite systems. Reviews of the history and extensive references are given in the papers by Crawford and Kino[16] and Fainberg[17] and also the monograph by Briggs.[18] The papers given in these references treat different aspects of the beam-plasma problem and in general do not consider the effects of magnetic fields, temperatures, collisions, and finite geometries as a combined analysis. Crawford[19] has considered the beam-plasma interaction process and has included temperature in the plasma, plasma collisions, and boundedness of the electron beam in a quasi-static approach with no d.c. magnetic field. Other investigations of beam-plasma interaction have recently been carried out by the Plasma Research Group at Stanford.[20]

The purpose of this investigation is to develop a theoretical model that includes the effects of temperatures, collisions, external d.c. magnetic fields, and finite dimension of the system without using the quasi-static approach in describing the electromagnetic field.

The analysis that is developed does not consider gradients in the d.c. electron densities or gradients of beam velocity and is based on a linearization of the hydrodynamic equations and Maxwell's field equations. The development is a small signal analysis and is not valid for large signal perturbations of the d.c. quantities. The theory is valid for small d.c. magnetic fields, but would not be expected to hold when the pressures become non-diagonal tensor quantities due to the effects of the d.c. magnetic field. In this case, the forces due to the pressure terms would have to be written as a divergence of a pressure tensor rather than a gradient as is shown in the development of Chapter 2. In the limit of an infinite magnetic field, however, the equations describing the system can be obtained by taking the limit of large magnetic fields since, in this case, the pressures can be treated as scalar quantities which have variation only along the magnetic field direction.

In Chapter 2, the equations are developed by linearizing the original fluid and electromagnetic equations and are shown to yield a consistent mathematical description. The potential functions from which the transverse quantities are obtained are derived and the relations between the transverse components and the potentials are given.

Chapter 3 considers the unbounded beam-plasma system and the dispersion relations for both transverse $(\nabla \cdot \vec{E} = 0)$ and longitudinal waves $(\nabla \times \vec{E} = 0)$. Graphs of real and imaginary parts of the propagation constant along the direction of the beam are given for different values of collision frequency, temperature, electron density, and beam velocity for real excitation frequency, ω.

The general bounded system is considered in Chapter 4 where the method used to decouple the potential equations is explained for the regions containing the plasma and beam and for the region containing only the plasma.

Chapter 5 describes a particular example, that of the infinite plasma through which an electron beam of finite radius passes. The solutions for the potential functions are exhibited in terms of Bessel functions and Hankel functions and the boundary conditions placed on these potential functions yield the allowable eigenvalues and propagation constants for the problem. In this section, only the solutions which have wave vector components in the direction of the d.c. beam velocity are computed and the graphical results for the real and imaginary parts of the propagation constant for real values of frequency, ω , are displayed. It can be seen from a comparison of these results and the results given for the unbounded system, that the gain curves are considerably less broadband for the finite radius beam case and do not indicate as high a gain value.

Although the question of stability in this problem is an important one, no analysis of the system stability is developed. Self[21] has shown that collision frequencies of a very small magnitude tend to cancel the effects of "absolute" instabilities which grow in time and therefore it is expected that the traveling-wave instability would be dominant for propagation in the direction of the d.c. electron beam velocity.

Chapter 2

THEORETICAL MODEL

2.1 Introduction

The two-fluid model chosen for this investigation of the beam-plasma interaction system is based on a linearization of the fluid transport equations, the adiabatic equation of state, and Maxwell's field equations. The electron beam and plasma are assumed to occupy the same region of space so that the most general formulation of equations is developed. It is also assumed that the system is excited by a high frequency source and the a.c. motion of the ions can be neglected.

The cold ions provide the positive background charge required to maintain a macroscopically neutral steady-state plasma and beam system. The effects of finite temperatures of the beam and plasma electrons are introduced through the adiabatic equation of state and the perfect gas law. Electron-neutral and electron-ion collisions are included by introducing effective collision frequencies in the force equations.

No attempt has been made to consider inhomogeneities of the d.c. plasma or gradients in the d.c. beam velocity. The effects of electron-electron collisions are included only as they give rise to their own species electron pressure term and effects due to ionization, recombination, and attachment are not included explicitly.

In the development which follows, no quasi-static approximation has been made which neglects the a.c. magneitc field as is usually done in the treatment of this problem. It will be shown that keeping terms of the same order in the linearization, including the a.c. magnetic field terms, leads to a set of self-consistent equations with the electric and magnetic fields satisfying Maxwell's constitutive equations. Although it might appear reasonable to neglect the a.c. magnetic field from a comparison with the electric field, it is instructive to carry along the a.c. magnetic field terms. This is particularly true in consideration of terms arising in the a.c. power theorems (Appendix A) and the dispersion equations for transverse waves ($\nabla \cdot \vec{E} = 0$) where terms arising from the a.c. magnetic field are of the same order as terms of the quasi-static results.

The description of all quantities is referred to the fixed laboratory frame and it is assumed that the electron beam that traverses the plasma is nonrelativistic. The description of the beam quantities in the fixed reference frame is obtained by using the Lorentz transformation of the electric field and the Doppler shifted frequency.

2.2 Development of Linearized Equations

The system of equations to be linearized is of the form:

Maxwell's Equations

$$\nabla \times \vec{E} = - \mu_o \frac{\partial \vec{H}}{\partial t} - \vec{J}_m \tag{2-1}$$

$$\nabla \times \vec{H} = \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \vec{J}_e \tag{2-2}$$

Force Equation

$$m\,n\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) - q\,n(\vec{E} + \vec{v} \times \vec{B}) + \nabla P + m\,n \sum_{\beta} \nu_{q\beta}(\vec{v} - \vec{v}_{\beta}) = \vec{F} \tag{2-3}$$

Continuity Equation

$$\nabla \cdot (n\,\vec{v}) + \frac{\partial n}{\partial t} = S \tag{2-4}$$

Polytropic State Equation

$$P\,n^{-\gamma} = \text{Constant} \tag{2-5}$$

Perfect Gas Law

$$P = n\,K\,T \tag{2-6}$$

where

$\vec{E}$ = electric field intensity

$\vec{H}$ = magnetic field intensity

μ_o = permeability of free space

ε_o = permittivity of free space

$\vec{J}_e$ = electric current density source

$\vec{J}_m$ = magnetic current density source

$\vec{F}$ = externally applied forcing function

S = particle sink (source) function

q = charge of particles under consideration

n = density of charged particles

$\vec{v}$ = charged particle velocity

$\vec{B} = \mu_o \vec{H}$

P = pressure of charged particles due to thermal motion

T = temperature of charged particles

K = Boltzmann's constant

γ = compression constant which for adiabatic conditions is the ratio of specific heat at constant pressure to specific heat at constant volume

$\vec{v}_\beta$ = velocity of particles that collide with the charged particles under consideration

$\nu_{q\beta}$ = effective collision frequency for momentum transfer of the charged particles of velocity $\vec{v}$ with particles of velocity $\vec{v}_\beta$.

In obtaining the collision term for Eq. (2-3), it was assumed that collisions of particles of charge q and velocity $\vec{v}$ with particles of species β is such that the momentum transferred is proportional to the relative velocities before collision, that is, the species β is assumed to have essentially an infinite mass and the effective collision frequency gives an average measure of the momentum transfer as if all collisions were "head on". This approximation assumes collisions of charged particles with neutrals or ions of much larger mass than that of the charged particles under consideration. Thus, scattering collisions which would give rise to the temperature of the charged species are not considered separately in the collision frequency term. The randomness of motion is accounted for in the pressure term of the force equation.

A linearization of the set of equations, (2-1)-(2-6), is obtained by using the assumption that each variable is composed of a steady and a time varying part. Thus, each term is written:

$$\vec{E} = \vec{E}_o + \vec{E}_1 \tag{2-7a}$$

$$\vec{H} = \vec{H}_o + \vec{H}_1 \tag{2-7b}$$

$$\vec{v} = \vec{v}_o + \vec{v}_1 \tag{2-7c}$$

$$\vec{v}_\beta = \vec{v}_{\beta o} + \vec{v}_{\beta 1} \tag{2-7d}$$

$$n = n_o + n_1 \tag{2-7e}$$

$$P = P_o + P_1 \tag{2-7f}$$

$$T = T_o + T_1 \tag{2-7g}$$

$$\vec{J}_e = \vec{J}_{eo} + \vec{J}_{e1} \tag{2-7h}$$

$$\vec{J}_m = \vec{J}_{mo} + \vec{J}_{m1} \tag{2-7i}$$

$$S = S_o + S_1 \tag{2-7j}$$

$$\vec{F} = \vec{F}_o + \vec{F}_1 \tag{2-7k}$$

where the (o) subscript refers to the steady, time invariant quantities and the (1) subscript refers to the quantities which have time variations. Substitution of these quantities into Eqs. (2-1)-(2-6) yields two sets of equations when terms of the zeroth and first order are equated separately.

$$\nabla \times \vec{E}_1 = -\mu_o \frac{\partial \vec{H}_1}{\partial t} - \vec{J}_{m1} \tag{2-8a}$$

$$\nabla \times \vec{E}_o = -\vec{J}_{mo} \tag{2-8b}$$

$$\nabla \times \vec{H}_1 = \varepsilon_o \frac{\partial \vec{E}_1}{\partial t} + q(n_o\, \vec{v}_1 + n_1\, \vec{v}_o) + \vec{J}_{e1} \tag{2-9a}$$

$$\nabla \times \vec{H}_o = q\, n_o\, \vec{v}_o + \vec{J}_{eo} \tag{2-9b}$$

$$\begin{aligned} m\, n_o\left(\frac{\partial \vec{v}_1}{\partial t} + \{\vec{v}_o \cdot \nabla\}\, \vec{v}_1 + \{\vec{v}_1 \cdot \nabla\}\, \vec{v}_o\right) &+ m\, n_1\{\vec{v}_o \cdot \nabla\}\, \vec{v}_o \\ = q\, n_o(\vec{E}_1 + \vec{v}_1 \times \mu_o\, \vec{H}_o + \vec{v}_o \times \mu_o\, \vec{H}_1) &+ q n_1(\vec{E}_o + \vec{v}_o \times \mu_o\, \vec{H}_o) \\ - \nabla P_1 - m\, n_o \sum_\beta \nu_{q\beta}(\vec{v}_1 - \vec{v}_{\beta 1}) & \\ - m\, n_1 \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_1 & \end{aligned} \tag{2-10a}$$

$$\begin{aligned} m\, n_o\{\vec{v}_o \cdot \nabla\}\, \vec{v}_o = q\, n_o(\vec{E}_o + \vec{v}_o \times \mu_o\, \vec{H}_o) - \nabla P_o & \\ - m\, n_o \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_o & \end{aligned} \tag{2-10b}$$

$$\begin{aligned} n_o\, \nabla \cdot \vec{v}_1 + \{\vec{v}_o \cdot \nabla\}\, n_1 + n_1\, \nabla \cdot \vec{v}_o + \{\vec{v}_1 \cdot \nabla\}\, n_o & \\ = -\frac{\partial n_1}{\partial t} + S_1 & \end{aligned} \tag{2-11a}$$

$$n_o\, \nabla \cdot \vec{v}_o + \{\vec{v}_o \cdot \nabla\}\, n_o = S_o \tag{2-11b}$$

$$P_1 = \frac{\gamma\, P_o\, n_1}{n_o} \tag{2-12a}$$

$$P_o \, n_o^{-\gamma} = \text{Constant} \tag{2-12b}$$

$$P_1 = K(n_1 \, T_o + n_o \, T_1) \tag{2-13a}$$

$$P_o = n_o \, K \, T_o \tag{2-13b}$$

In Eqs. (2-10a,b), the collision term was split so that the collision frequency $\nu_{q\beta}^{o}$ for the steady velocities can be different from that describing collisions for the time dependent velocities. One would expect that the collision frequencies would be different on a physical basis since the cross section for collision is velocity dependent. Also, Eqs. (2-12a,b) are obtained from Eq. (2-5) by using the first two terms of the binomial expansion. The convection currents in Eqs. (2-9a,b) are written explicitly in terms of densities and velocities and only external current sources are contained in $\vec{J}_{eo,1}$.

If we make the additional assumption that the steady variables $\vec{v}_o$, P_o , and n_o have no spatial variation, we obtain the following set of equations:

$$\nabla \times \vec{E}_1 = -\mu_o \frac{\partial \vec{H}_1}{\partial t} - \vec{J}_{m1} \tag{2-13a}$$

$$\nabla \times \vec{E}_o = -\vec{J}_{mo} \tag{2-13b}$$

$$\nabla \times \vec{H}_1 = \varepsilon_o \frac{\partial \vec{E}_1}{\partial t} + q(n_o \, \vec{v}_1 + n_1 \, \vec{v}_o) + \vec{J}_{e1} \tag{2-14a}$$

$$\nabla \times \vec{H}_o = q\, n_o\, \vec{v}_o + \vec{J}_{eo} \tag{2-14b}$$

$$m\, n_o\left(\frac{\partial \vec{v}_1}{\partial t} + \{\vec{v}_o \cdot \nabla\}\, \vec{v}_1\right) = q\, n_o(\vec{E}_1 + \vec{v}_1 \times \mu_o\, \vec{H}_o + \vec{v}_o \times \mu_o\, \vec{H}_1)$$

$$+ q\, n_1(\vec{E}_o + \vec{v}_o \times \mu_o\, \vec{H}_o) - \nabla P_1 - m\, n_o \sum_\beta \nu_{q\beta}(\vec{v}_1 - \vec{v}_{\beta 1}) \tag{2-15a}$$

$$- m\, n_1 \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_1$$

$$0 = q\, n_o(\vec{E}_o + \vec{v}_o \times \mu_o\, \vec{H}_o) - m\, n_o \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_o \tag{2-15b}$$

$$n_o\, \nabla \cdot \vec{v}_1 + \{\vec{v}_o \cdot \nabla\}\, n_1 = - \frac{\partial n_1}{\partial t} + S_1 \tag{2-16a}$$

$$0 = S_o \tag{2-16b}$$

$$P_1 = \frac{\gamma\, P_o\, n_1}{n_o} \tag{2-17a}$$

$$P_o\, n_o^{-\gamma} = \text{Constant} \tag{2-17b}$$

$$P_1 = K(n_1\, T_o + n_o\, T_1) \tag{2-18a}$$

$$P_o = n_o\, K\, T_o \tag{2-18b}$$

Substituting Eq. (2-15b), multiplied by n_1/n_o , into Eq. (2-15a) gives the first order force equation:

$$m\, n_o\left(\frac{\partial \vec{v}_1}{\partial t} + \{\vec{v}_o \cdot \nabla\}\, \vec{v}_1\right) = q\, n_o(\vec{E}_1 + \vec{v}_1 \times \mu_o\, \vec{H}_o + \vec{v}_o \times \mu_o\, \vec{H}_1)$$

$$- \nabla P_1 - m\, n_o \sum_\beta \nu_{q\beta}(\vec{v}_1 - \vec{v}_{\beta 1}) + \vec{F}_1 - \frac{n_1}{n_o}\vec{F}_o \tag{2-19}$$

The remaining first order equations are written:

$$\nabla \times \vec{E}_1 = -\mu_o \frac{\partial \vec{H}_1}{\partial t} - \vec{J}_{m1} \tag{2-20}$$

$$\nabla \times \vec{H}_1 = \varepsilon_o \frac{\partial \vec{E}_1}{\partial t} + q(n_o\, \vec{v}_1 + n_1\, \vec{v}_o) + \vec{J}_{e1} \tag{2-21}$$

$$n_o\, \nabla \cdot \vec{v}_1 + \{\vec{v}_o \cdot \nabla\}\, n_1 = -\frac{\partial n_1}{\partial t} + S_1 \tag{2-22}$$

$$P_1 = \frac{\gamma\, P_o\, n_1}{n_o} \tag{2-23}$$

$$P_1 = K(n_1\, T_o + n_o\, T_1) \tag{2-24}$$

Similarly, the zeroth order set is written:

$$\nabla \times \vec{E}_o = -\vec{J}_{mo} \tag{2-25}$$

$$\nabla \times \vec{H}_o = q\, n_o\, \vec{v}_o + \vec{J}_{eo} \tag{2-26}$$

$$0 = q\, n_o(\vec{E}_o + \vec{v}_o \times \mu_o\, \vec{H}_o) - m\, n_o \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_o \tag{2-27}$$

$$0 = S_o \tag{2-28}$$

$$P_o\, n_o^{-\gamma} = \text{Constant} \tag{2-29}$$

$$P_o = n_o\, K\, T_o \tag{2-30}$$

We now define a thermal sound speed, u , such that

$$u^2 = \frac{\gamma\ KT_o}{m} \quad . \tag{2-31}$$

Using Eqs. (2-23) and (2-30) to relate n_1 and P_1, we can put the first order set of equations in the final form:

$$\nabla \times \vec{E}_1 = -\mu_o \frac{\partial \vec{H}_1}{\partial t} - \vec{J}_{m1} \tag{2-32}$$

$$\nabla \times \vec{H}_1 = -\varepsilon_o \frac{\partial \vec{E}_1}{\partial t} + q\left(n_o\ \vec{v}_1 + \frac{P_1}{m\ u^2}\ \vec{v}_o\right) + \vec{J}_{e1} \tag{2-33}$$

$$m\ n_o\left(\frac{\partial \vec{v}_1}{\partial t} + \{\vec{v}_o \cdot \nabla\}\ \vec{v}_1\right) = q\ n_o(\vec{E}_1 + \vec{v}_1 \times \mu_o\ \vec{H}_o + \vec{v}_o \times \mu_o\ \vec{H}_1)$$

$$- \nabla P_1 - m\ n_o \sum_\beta \nu_{q\beta}(\vec{v}_1 - \vec{v}_{\beta 1}) + \vec{F}_1 - \frac{P_1}{mu^2\ n_o}\ \vec{F}_o \tag{2-34}$$

$$n_o\ mu^2\ \nabla \cdot \vec{v}_1 + \{\vec{v}_o \cdot \nabla\}\ P_1 = -\frac{\partial P_1}{\partial t} + m\ u^2\ S_1 \tag{2-35}$$

The zeroth order set remains:

$$\nabla \times \vec{E}_o = -\vec{J}_{mo} \tag{2-36}$$

$$\nabla \times \vec{H}_o = q\ n_o\ \vec{v}_o + \vec{J}_{eo} \tag{2-37}$$

$$0 = q\ n_o(\vec{E}_o + \vec{v}_o \times \mu_o\ \vec{H}_o) - m\ n_o \sum_\beta \nu^o_{q\beta}(\vec{v}_o - \vec{v}_{\beta o}) + \vec{F}_o \tag{2-38}$$

$$0 = S_o \tag{2-39}$$

The fact that Eq. (2-39) forces the problem to have no constant sources or sinks which produce charged particles of charge q and mass m is a direct result of the restrictions that n_o, $\vec{v}_o$, and P_o have no spatial variation.

2.3 Electron Beam-Plasma System Equations

In developing the equations for a combined beam-plasma system, each species of electron (e.g., beam electron) must satisfy, separately, a force equation and a continuity equation and it is assumed that each species is governed by a polytropic equation of state ($Pn^{-\gamma}$ = Constant) and the perfect gas law. This is equivalent to saying that each electron of the combined system can be identified at any time and position as being associated with the beam or plasma. Maxwell's field equations then couple the two sets of fluid equations through the electric and magnetic fields associated with the electron motion.

The purpose of this investigation is to consider wave propagation in a beam-plasma system and thus it will be assumed that all field quantities can be written as explicit functions of time and the coordinate along the beam direction. We shall also assume that all field quantities can be expressed as:

$$Q_1(\vec{r},t) = Q_1(\vec{r}_t)\, e^{j(\omega t - kz)}, \qquad (2\text{-}40)$$

where $\vec{r}$ is a three dimensional position vector, ω is the radian excitation frequency of the wave, z is the direction along the beam, $\vec{r}_t$ is a vector describing the coordinates transverse to z, k is the propagation constant in the z direction, and t and j are time and the square root of minus one respectively. The actual physical quantities are obtained by taking the real or imaginary part of Eq. (2-40), thus:

$$Q_1(\vec{r},t) = \mathrm{Re}\{Q_1(\vec{r}_t)\, e^{j(\omega t-kz)}\} \quad \text{or} \quad \mathrm{Im}\{Q_1(\vec{r}_t)\, e^{j(\omega t-kz)}\} \quad . \tag{2-41}$$

The time and space dependence can also be interpreted in a different manner if it is assumed that double-sided Laplace transforms have been taken so that

$$Q_1(\vec{r}_t,\omega,k) = \int_{z=-\infty}^{\infty} \int_{t=-\infty}^{\infty} Q_1(\vec{r},t)\, e^{j(\omega t-kz)}\, dt\, dz \quad . \tag{2-42}$$

The time and space dependence, is recovered by the inversion integrals which are taken along appropriate contours.

The first interpretation will be used here since the purpose of this investigation is to study discrete propagating waves excited at a single real frequency, ω . Whether the first or second formulation is adopted, however, the derivatives with respect to time and the z dimension appear as:

$$\frac{\partial Q_1}{\partial t} \to j\omega\, Q_1 \quad , \quad \frac{\partial Q_1}{\partial z} \to -\, jk\, Q_1 \tag{2-43}$$

for all of the first order field variables and therefore, the results derived in this paper apply equally well for real wave solutions or solutions in the transformed space.

The equations of the first order describing the beam-plasma system are written here with the explicit dependence on z and t being suppressed since the same term multiplies all variables of the first order. Thus, we arrive at the final form for the first order equations describing the beam-plasma system where the subscript (1) is implied but has been deleted from all first order terms:

$$\nabla \times \vec{E} + j\omega\, \mu_o\, \vec{H} = - \vec{J}_m \tag{2-44}$$

$$\nabla \times \vec{H} - j\omega\, \varepsilon_o\, \vec{E} + e(N_p\, \vec{v}_p + N_b\, \vec{v}_b + \frac{P_b}{m\, u_b^2} \vec{V}_b) = \vec{J}_e \tag{2-45}$$

$$m\, N_p(j\omega + \nu_p)\vec{v}_p + N_{pe}(\vec{E} + \vec{v}_p \times \mu_o \vec{H}_o) + \nabla P_p = \vec{F}_p - \frac{P_p}{m\, u_p^2\, N_p} \vec{F}_{po} \tag{2-46}$$

$$m\, N_b(j\{\omega - kV_b\} + \nu_b)\vec{v}_b + N_b e(\vec{E} + \vec{v}_b \times \mu_o \vec{H}_o + \vec{V}_b \times \mu_o \vec{H})$$

$$+ \nabla P_b = \vec{F}_b - \frac{P_b}{m\, u_b^2\, N_b} \vec{F}_{bo} \tag{2-47}$$

$$u_p^2\, m\, N_p\, \nabla \cdot \vec{v}_p + j\omega\, P_p = S_p \tag{2-48}$$

$$u_b^2\, m\, N_b\, \nabla \cdot \vec{v}_b + j(\omega - k\, V_b)\, P_b = S_b \quad . \tag{2-49}$$

The subscripts b,p refer to the beam and plasma properties respectively. In this final form, the variables are defined:

$N_{b,p}$ = d.c. space and time invariant electron densities

$\vec{V}_b$ = d.c. electron beam velocity

$\vec{H}_o$ = d.c. magnetic field intensity

$\vec{E},\vec{H}$ = a.c. electric and magnetic field intensities respectively

$\vec{v}_{b,p}$ = a.c. electron velocities

$P_{b,p}$ = a.c. electron pressures

= $n_{b,p}\, m u^2_{b,p}$ where $n_{b,p}$ are the a.c. electron density variations and $u^2_{b,p} = \frac{\gamma KT}{m}{}_{p,bo}$ are the squares of the thermal sound speeds

ω = radian excitation frequency

e = magnitude of the electronic charge

m = mass of the electron

$\nu_{b,p}$ = effective collision frequencies for electron-ion and electron-neutral collisions

k = propagation constant in the z direction

μ_o = permeability of free space

ϵ_o = permittivity of free space

$\vec{J}_m$ = a.c. magnetic source current

$\vec{J}_e$ = a.c. electric current source

$S_{b,p}$ = a.c. electron sinks (sources) for beam and plasma electrons

$\vec{F}_{b,p}$ = external a.c. forces for beam and plasma electrons

$\vec{F}_{b,po}$ = d.c. forcing function for beam and plasma electrons respectively.

The collision terms in the preceeding equations were written assuming that the heavy particles (e.g., neutrals, ions) are stationary so that momentum change is directly proportional to the electron velocity. Thus, $\nu_{b,p}$ are effective collision frequencies for the combined electron-neutral, electron-ion collisions assuming that the neutrals and ions have no motion.

The d.c. equations are now written as follows:

$$\nabla \times \vec{E}_o = - \vec{J}_{mo} \tag{2-50}$$

$$\nabla \times \vec{H}_o = - e\, N_b\, \vec{V}_b + \vec{J}_{eo} \tag{2-51}$$

$$0 = - e\, N_p\, \vec{E}_o + \vec{F}_{po} \tag{2-52}$$

$$0 = S_{po} \tag{2-53}$$

$$0 = - eN_b(\vec{E}_o + \vec{V}_b \times \mu_o\, \vec{H}_o) - m\, N_b\, \nu^o\, \vec{V}_b + \vec{F}_{bo} \tag{2-54}$$

$$0 = S_{bo} \tag{2-55}$$

where

$\vec{E}_o$ = d.c. electric field intensity

$\vec{J}_{mo}$ = d.c. magnetic current source

$\vec{J}_{eo}$ = d.c. electric current source

ν^o = collision frequency of the d.c. beam for steady momentum transfer

$\vec{H}_o$ = d.c. magnetic field intensity

$S_{b,p}$ = d.c. electron sinks (sources) for the beam and plasma electrons

$\vec{F}_{b,po}$ = d.c. beam and plasma force functions.

The restrictions imposed on the steady variables such as the condition that S_{bo} and S_{po} be zero and the relations between the electric field and the force functions $\vec{F}_{bo}$ and $\vec{F}_{po}$ are a direct result of assuming that the beam velocity $\vec{V}_b$ and the densities N_b , N_p be constant in space and time. Thus, to have a completely consistent system, we must apply or induce no constant electric or magnetic fields unless they satisfy the Eqs. (2-50)-(2-55), and no sources or sinks of electrons can act on the system if they are constant in time. It is also clear that $\vec{H}_o$ is, in general, not independent of position, since part of the steady magnetic field is caused by the steady beam current. However, currents in most beam problems are very small or external magnetic fields are applied so that the effect of this induced magnetic field can easily be neglected. In all of the development that follows, the assumption will be made that this induced magnetic field term can be neglected in the first order equations and that any d.c. magnetic field is constant in space as well as time.

2.4 Transverse and Longitudinal Field Separation

We now separate the vector field quantities describing the beam-plasma system into components along the beam (longitudinal) and transverse to the beam direction. The general vector field quantity is written as follows:

$$\vec{Q}(\vec{r}_t,k,\omega) = Q_t(\vec{r}_t,k,\omega)\,\vec{a}_t + Q_z(\vec{r}_t,k,\omega)\,\vec{a}_z \quad , \tag{2-56}$$

where $\vec{a}_t$ and $\vec{a}_z$ are unit vectors transverse to and along the beam, respectively, and the subscripts t,z refer to transverse and z directed quantities. By separating the del operator, ∇ in the form

$$\nabla = \{\nabla_t + \vec{a}_z \frac{\partial}{\partial z}\} = \{\nabla_t - jk\,\vec{a}_z\} \quad , \tag{2-57}$$

we can write the following relations:

$$\nabla \times \vec{A} = \nabla_t \times \vec{A}_t - \vec{a}_z \times \nabla_t A_z - jk\,\vec{a}_z \times \vec{A}_t \tag{2-58}$$

$$\nabla \cdot \vec{A} = \nabla_t \cdot \vec{A}_t - jk\,A_z \tag{2-59}$$

$$\nabla \phi = \nabla_t \phi - jk\,\vec{a}_z\,\phi \tag{2-60}$$

$$\nabla_t \times (\vec{a}_z \times \vec{A}_t) = \vec{a}_z(\nabla_t \cdot \vec{A}_t) \tag{2-61}$$

$$\nabla_t \cdot (\vec{A}_t \times \vec{a}_z) = \vec{a}_z \cdot (\nabla_t \times \vec{A}_t) \tag{2-62}$$

The beam-plasma set of equations, (2-44)-(2-49), is now written in the separated form by using the above relations.

$$-\vec{a}_z \times \nabla_t E_z - jk\,\vec{a}_z \times \vec{E}_t + j\omega\,\mu_o\,\vec{H}_t = -\vec{J}_{mt} \tag{2-63}$$

$$\nabla_t \times \vec{E}_t + j\omega\,\mu_o\,\vec{H}_t = -\vec{J}_{mz} \tag{2-64}$$

$$-\vec{a}_z \times \nabla_t H_z - jk\,\vec{a}_z \times \vec{H}_t - j\omega\,\varepsilon_o\,\vec{E}_t + e(N_p\vec{v}_{pt} + N_b\vec{v}_{bt}) = \vec{J}_{et} \tag{2-65}$$

$$\nabla_t \times \vec{H}_t - j\omega\,\varepsilon_o\,\vec{E}_z + e\left(N_p\vec{v}_{pz} + N_b\vec{v}_{bz} + \frac{P_b\,\vec{V}_b}{mu_b^2}\right) = \vec{J}_{ez} \tag{2-66}$$

$$(j\omega + \nu_p)m\,N_p\,\vec{v}_{pt} + N_p e(\vec{E}_t + \vec{v}_{pt} \times \vec{B}_o) + \nabla_t\,P_p$$

$$= \vec{F}_{pt} - \frac{P_p}{m\,u_p^2\,N_p}\,\vec{F}_{pot} \tag{2-67}$$

$$(j\omega + \nu_p)m\,N_p\,\vec{v}_{pz} + N_p e\,\vec{E}_z - jk\,P_p\,\vec{a}_z = \vec{F}_{pz} - \frac{P_p}{m\,u_p^2\,N_p}\,\vec{F}_{poz} \tag{2-68}$$

$$(j\{\omega - kV_b\} + \nu_b)m\,N_b\,\vec{v}_{bt} + N_b e(\vec{E}_t + \vec{v}_{bt} \times \vec{B}_o + \vec{V}_b \times \mu_o\,\vec{H}_t)$$

$$+ \nabla_t\,P_b = \vec{F}_{bt} - \frac{P_b}{m\,u_b^2\,N_b}\,\vec{F}_{bot} \tag{2-69}$$

$$(j\{\omega - kV_b\} + \nu_b)m\,N_b\,\vec{v}_{bz} + N_b e\,\vec{E}_z - jk\,P_b\,\vec{a}_z$$

$$= \vec{F}_{bz} - \frac{P_b}{m\,u_b^2\,N_b}\,\vec{F}_{boz} \tag{2-70}$$

$$u_p^2\,m\,N_p(\nabla_t \cdot \vec{v}_{pt} - jk\,v_{pz}) + j\omega\,P_p = S_p \tag{2-71}$$

$$u_b^2 \, m \, N_b(\nabla_t \cdot \vec{v}_{bt} - jk \, v_{bz}) + j(\omega - kV_b) \, P_b = S_b \tag{2-72}$$

2.5 Potential Function Equations

We can derive a set of coupled second order differential equations involving potential functions from which the total field solution is obtained. In this regard, we shall pick as potential functions, the z directed quantities E_z, H_z, and the two pressures P_p and P_b. If a relationship between these potential functions and the other field quantities is found such that all field quantities can be derived from these potential functions alone and the potential functions are unique for the given boundary conditions, then, we have obtained a unique solution to the beam-plasma interaction problem.

The derivation of the potential function equations is long and tedious and will not be presented here, but is presented in Appendix B. Only the results of the derivation for the sourceless or homogeneous equations are given here.

$$\begin{aligned} \nabla_t^2 E_z &+ \left\{ k_o^2 \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)} \right) - k^2 \right\} E_z \\ &+ \left\{ \frac{jk \, e}{m \, \varepsilon_o} \left(\frac{\omega \, \mu_o \, \varepsilon_o}{\omega - j\nu_p} - \frac{1}{u_p^2} \right) \right\} P_p \\ &+ \left\{ \frac{jk \, e}{m \, \varepsilon_o} \left(\frac{\omega \, \mu_o \, \varepsilon_o}{(\omega - kV_b - j\nu_b)} - \frac{1}{u_b^2} \right) + j \, \frac{\omega \, \mu_o \, e \, V_b}{m u_b^2} \right\} P_b = 0 \end{aligned} \tag{2-73}$$

$$\nabla_t^2 H_z + \left\{ k_o^2 \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2(\omega - kV_b)}{\omega^2(\omega - kV_b - j\nu_b)} \right) - k^2 \right\} H_z$$

$$- jk\,\varepsilon_o \Omega \left\{ \frac{\omega_p^2}{(\omega - j\nu_p)^2} + \frac{\omega_b^2}{(\omega - kV_b - j\nu_b)^2} \right\} E_z$$

$$+ \frac{\left\{ e\Omega \left(\omega - \dfrac{k^2\, u_p^2}{(\omega - j\nu_p)} \right) \right\} P_p}{(\omega - j\nu_p)\, m\, u_p^2} \qquad (2\text{-}74)$$

$$+ \frac{\left\{ e\Omega \left(\omega - kV_b - \dfrac{k^2\, u_b^2}{(\omega - kV_b - j\nu_b)} \right) \right\} P_b}{(\omega - kV_b - j\nu_b)\, m\, u_b^2} = 0$$

$$\nabla_t^2 P_p + \frac{1}{u_p^2} \left\{ \omega(\omega - j\nu_p) - k^2 u_p^2 - \omega_p^2 - \frac{\Omega^2}{(\omega - j\nu_p)^2} \left[\omega(\omega - j\nu_p) - k^2 u_p^2 \right] \right\} P_p$$

$$- \frac{\omega_p^2}{u_b^2} P_b + \frac{m}{e} \Omega \left\{ \frac{\omega\, \mu_o\, \varepsilon_o\, \omega_p^2}{\omega - j\, \nu_p} \right\} H_z + \left\{ \frac{jk\, \varepsilon_o\, \Omega^2\, \omega_p^2 \frac{m}{e}}{(\omega - j\, \nu_p)^2} \right\} E_z = 0 \qquad (2\text{-}75)$$

$$\nabla_t^2 P_b + \frac{1}{u_b^2} \left\{ (\omega - kV_b)(\omega - kV_b - j\nu_b) - k^2 u_b^2 - \omega_b^2 - \Omega^2 \frac{\left[(\omega - kV_b)(\omega - kV_b - j\nu_b) - k^2 u_b^2 \right]}{(\omega - kV_b - j\nu_b)^2} \right.$$

$$\left. + \omega_b^2\, V_b\, u_o \varepsilon_o \left[\frac{k\, u_b^2}{(\omega - kV_b - j\nu_b)} + V_b \right] \right\} P_b + \left\{ \frac{\omega_b^2\, kV_b\, \mu_o \varepsilon_o}{(\omega - j\nu_p)} - \frac{\omega_b^2}{u_p^2} \right\} P_p \qquad (2\text{-}76)$$

$$+ j \left\{ \frac{k\Omega^2 \frac{m}{e} \omega_b^2\, \varepsilon_o}{(\omega - kV_b - j\nu_b)^2} - \omega_b^2 \frac{m}{e} V_b\, u_o \varepsilon_o^2\, \omega \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)} \right) \right\} E_z$$

$$+ \left\{ \frac{(\omega - kV_b)\, \omega_b^2\, \mu_o\, \varepsilon_o \frac{m}{e} \Omega}{(\omega - kV_b - j\nu_b)} \right\} H_z = 0$$

We have yet to derive the equations from which the transverse field quantities can be obtained from the potential functions E_z , H_z , P_p , P_b . This is the purpose of the next section.

2.6 Transverse Field-Potential Relationships

From Eq. (2-67) with all the external sources placed equal to zero, we obtain the relationship for $\vec{v}_{pt}$:

$$(j\omega + \nu_p)m\, N_p\, \vec{v}_{pt} - N_p e\, B_o\, \vec{a}_z \times \vec{v}_{pt} + N_p e\, \vec{E}_t + \nabla_t\, P_p = 0 \quad . \qquad (2\text{-}77)$$

We can represent the cross product for a right handed coordinate system as

$$\vec{a}_z \times \vec{v}_{pt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \left\{ \vec{v}_{pt} \right\} , \qquad (2\text{-}78)$$

where in Cartesian coordinates Eq. (2-78) becomes:

$$\vec{a}_z \times \vec{v}_{pt} = \vec{a}_z \times \{v_{px}\, \vec{a}_x + v_{py}\, \vec{a}_y\} = v_{px}\, \vec{a}_y - v_{py}\, \vec{a}_x$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{px} \\ v_{py} \end{bmatrix} = \begin{Bmatrix} -v_{py} \\ v_{px} \end{Bmatrix} . \qquad (2\text{-}79)$$

Eq. (2-77) can be written in the following form with the aid of Eq. (2-79):

$$\begin{bmatrix} (j\omega + \nu_p)m\,N_p & N_p e\,B_o \\ \\ -\,N_p e\,B_o & (j\omega + \nu_p)m\,N_p \end{bmatrix} \cdot \vec{v}_{pt} = -\,\nabla_t\,P_p - N_p e\,\vec{E}_t \tag{2-80}$$

or

$$\vec{v}_{pt} = \bar{\bar{A}}^{-1} \cdot \left\{ -\,\frac{\nabla_t\,P_p - N_p e\,\vec{E}_t}{m\,N_p} \right\} \tag{2-81}$$

where

$$\bar{\bar{A}} = \begin{bmatrix} (j\omega + \nu_p) & \Omega \\ \\ -\,\Omega & (j\omega + \nu_p) \end{bmatrix} \tag{2-82}$$

and

$$\Omega = \frac{e\,B_o}{m} \tag{2-83}$$

with $\bar{\bar{A}}^{-1}$ representing the inverse matrix of $\bar{\bar{A}}$ such that $\bar{\bar{A}} \cdot \bar{\bar{A}}^{-1} = \bar{\bar{I}}$, the unit diagonal matrix.

The relation for $\vec{v}_{bt}$ is obtained from (2-69).

$$\vec{v}_{bt} = \bar{\bar{B}}^{-1} \cdot \left\{ -\,\frac{\nabla_t\,P_b - N_b e\,\vec{E}_t - N_b e\,V_b\,\mu_o\,\vec{a}_z \times \vec{H}_t}{m\,N_b} \right\} \tag{2-84}$$

where

$$\bar{\bar{B}} = \begin{bmatrix} j(\omega - kV_b) + \nu_b & \Omega \\ \\ -\,\Omega & j(\omega - kV_b) + \nu_b \end{bmatrix} \tag{2-85}$$

From Eq. (2-63), the expression for $\vec{a}_z \times \vec{H}_t$ becomes:

$$\vec{a}_z \times \vec{H}_t = -\frac{1}{j\omega\,\mu_o}\{\nabla_t E_z + jk\,\vec{E}_t\} \quad . \tag{2-86}$$

Substituting this expression into Eq. (2-84) yields:

$$\vec{v}_{bt} = \bar{\bar{B}}^{-1} \cdot \frac{\{-\nabla_t P_b - N_b e(1 - \frac{kV_b}{\omega})\,\vec{E}_t - j\,\frac{N_b e\,V_b}{\omega}\nabla_t E_z\}}{m\,N_b} \quad . \tag{2-87}$$

The result for $\vec{E}_t$ is obtained from (2-63), (2-65), (2-81) and (2-87).

$$\vec{E}_t = \left\{ j\,\frac{(k^2 - k_o^2)}{\omega\,\mu_o\,\varepsilon_o}\bar{\bar{I}} - \omega_p^2\,\bar{\bar{A}}^{-1} - \omega_b^2(1 - \frac{kV_b}{\omega})\,\bar{\bar{B}}^{-1}\right\}^{-1}$$

$$\cdot \left\{ \vec{a}_z \times \frac{\nabla_t H_z}{\varepsilon_o} + (j\,\frac{\omega_b^2\,V_b}{\omega}\,\bar{\bar{B}}^{-1} - \frac{k}{\omega\,\mu_o\,\varepsilon_o}\bar{\bar{I}}) \cdot \nabla_t E_z \right. \tag{2-88}$$

$$\left. + \frac{e}{m\varepsilon_o}\bar{\bar{A}}^{-1} \cdot \nabla_t P_p + \frac{e}{m\varepsilon_o}\bar{\bar{B}}^{-1} \cdot \nabla_t P_b \right\}$$

The relation for $\vec{H}_t$ is obtained from Eq. (2-63) and is written:

$$\vec{H}_t = \frac{k}{\omega\mu_o}\vec{a}_z \times \vec{E}_t + \frac{1}{j\omega\,\mu_o}\vec{a}_z \times \nabla_t E_z \quad . \tag{2-89}$$

We have shown that all transverse field quantities can be derived from a knowledge of the z directed potentials E_z , H_z , and the pressure P_p and P_b and therefore, the total field solutions are uniquely determined as long as the potential functions are themselves unique. This uniqueness is proven in Appendix D.

Chapter 3

UNBOUNDED BEAM-PLASMA INTERACTION

3.1 Introduction

The unbounded beam-plasma system is studied here to obtain some of the gross properties of such a system. One can usually obtain some physical feeling or insight to a phenomenon of considerable complexity when the most simple model for the problem is investigated. It is with this idea in mind that we now investigate the infinite dimensional beam-plasma system where the plasma and beam occupy the same space. Collisions, temperatures, and an external d.c. magnetic field are assumed to be present. Because the system is unbounded, no boundary conditions are placed on the beam and plasma quantities and the solutions to the system equations for first order variations are plane waves which propagate in the medium.

It should be noted here that all d.c. quantities such as densities, pressures, beam velocity, and magnetic field are space independent so that the system equations derived in Chapter 2 are applicable. As a consequence of these assumptions and the assumption of an infinite system, we must consider the effects of the d.c. magnetic field caused by the electron beam since, for very large distances from the axis of the system, the magnetic field becomes very large.

One could postulate a drifting ion stream which travels with the d.c. electron beam so that this magnetic field is absent and only externally applied steady magnetic fields exist in the problem. Such a drifting stream would still maintain a macroscopically neutral plasma and would not interact with the high frequency field excitations.

In the following development, we shall adopt the assumption of a drifting ion stream to compensate for the magnetic field caused by the electron beam. Such an assumption is not necessary in the small, finite system because the small magnetic field caused by the d.c. electron beam can be neglected.

3.2 Mathematical Formulation

Since the system is infinite in all directions and is also uniform, we can assume that all first order field quantities are plane waves of the form:

$$Q_1(\vec{r},t) = Q_1 \, e^{j(\omega t - \vec{k}\cdot\vec{r})} \quad , \tag{3-1}$$

where $\vec{r}$ is the three dimensional position vector in the space, $\vec{k}$ is the propagation vector.

With the field dependence expressed as in Eq. (3-1), the differential operations are expressed as follows:

$$\frac{\partial Q_1}{\partial t} \rightarrow j\omega\, Q_1$$

$$\nabla \times \vec{Q}_1 \rightarrow -\, j\vec{k} \times \vec{Q}_1$$

$$\nabla \cdot \vec{Q}_1 \rightarrow -\, j\vec{k} \cdot \vec{Q}_1 \qquad (3\text{-}2)$$

$$\nabla Q_1 \rightarrow -\, j\vec{k}\, Q_1 \quad ,$$

where the factor $e^{j(\omega t - \vec{k}\cdot\vec{r})}$ is suppressed since it multiplies each first order term.

The set of first order, linearized, beam-plasma equations obtained in Chapter 2 are written in the following fashion:

$$\nabla \times \vec{E} + \mu_o \frac{\partial \vec{H}}{\partial t} = 0 \qquad (3\text{-}3)$$

$$\nabla \times \vec{H} + e\left\{N_p\, \vec{v}_p + N_b\, \vec{v}_b + \frac{P_b\, \vec{V}_b}{m\, u_b^2}\right\} - \varepsilon_o \frac{\partial \vec{E}}{\partial t} = 0 \qquad (3\text{-}4)$$

$$m\, N_p \frac{\partial \vec{v}_p}{\partial t} + e\, N_p(\vec{E} + \vec{v}_p \times \vec{B}_o) + m\, N_p\, \nu_p\, \vec{v}_p + \nabla P_p = 0 \qquad (3\text{-}5)$$

$$m\, N_b \left\{\frac{\partial \vec{v}_b}{\partial t} + \{\vec{V}_b \cdot \nabla\}\, \vec{v}_b\right\} + e\, N_b(\vec{E} + \vec{V}_b \times \mu_o\, \vec{H} + \vec{v}_b \times \vec{B}_o)$$
$$+\, m\, N_b\, \nu_b\, \vec{v}_b + \nabla P_b = 0 \qquad (3\text{-}6)$$

$$u_p^2\, m\, \nabla \cdot (N_p\, \vec{v}_p) + \frac{\partial P_p}{\partial t} = 0 \qquad (3\text{-}7)$$

$$u_b^2 \, m \, \nabla \cdot (N_b \, \vec{v}_b) + \nabla \cdot (P_b \, \vec{V}_b) + \frac{\partial P_b}{\partial t} = 0 \qquad (3\text{-}8)$$

Using the relations of (3-2), this set of equations becomes:

$$\vec{k} \times \vec{E} - \omega\mu_o \, \vec{H} = 0 \qquad (3\text{-}9)$$

$$\vec{k} \times \vec{H} + j \, e(N_p \, \vec{v}_p + N_b \, \vec{v}_b + \frac{P_b}{m \, u_b^2} \vec{V}_b) + \omega \, \varepsilon_o \, \vec{E} = 0 \qquad (3\text{-}10)$$

$$m \, N_p(\omega - j\nu_p) \, \vec{v}_p - j \, N_p \, e(\vec{E} + \vec{v}_p \times \vec{B}_o) - \vec{k} \, P_p = 0 \qquad (3\text{-}11)$$

$$m \, N_b\{\omega - \vec{V}_b \cdot \vec{k} - j\nu_b\}\vec{v}_b - j \, N_b e(\vec{E} + \vec{V}_b \times \mu_o\vec{H} + \vec{v}_b \times \vec{B}_o)$$
$$- \vec{k} \, P_b = 0 \qquad (3\text{-}12)$$

$$u_p^2 \, m \, N_p(\vec{k} \cdot \vec{v}_p) - \omega \, P_p = 0 \qquad (3\text{-}13)$$

$$u_b^2 \, m \, N_b(\vec{k} \cdot \vec{v}_b) - (\omega - \vec{V}_b \cdot \vec{k}) \, P_b = 0 \qquad (3\text{-}14)$$

Equation (3-13) can be solved for P_p , yielding

$$P_p = \frac{u_p^2 \, m \, N_p}{\omega} (\vec{k} \cdot \vec{v}_p) \quad . \qquad (3\text{-}15)$$

Similarly, from Eq. (3-14),

$$P_b = \frac{u_b^2 \, m \, N_b}{(\omega - \vec{V}_b \cdot \vec{k})} (\vec{k} \cdot \vec{v}_b) \quad . \qquad (3\text{-}16)$$

We now assume that the beam velocity, $\vec{V}_b$, and the d.c. magnetic field, $\vec{B}_o$, are directed along the positive z axis. With this restriction, we can simplify the cross product terms involving $\vec{V}_b$ and $\vec{B}_o$ by representing the cross product using tensor notation. The cross product of the unit vector $\vec{a}_z$ with any arbitrary vector $\vec{F}$ is written

$$\vec{a}_z \times \vec{F} \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{3-17}$$

for a right-handed Cartesian coordinate system.

The expression for $\vec{v}_p$ is obtained from Eqs. (3-11) and (3-15) with the aid of Eq. (3-17)

$$\vec{v}_p = j\,\frac{e}{m}\left(\overline{\overline{C}} - \vec{k}\,\vec{k}\,\frac{u_p^2}{\omega}\right)^{-1} \cdot \vec{E} \tag{3-18}$$

where the tensors $\overline{\overline{C}}$ and $\vec{k}\,\vec{k}$ are defined as:

$$\overline{\overline{C}} \equiv \begin{bmatrix} \omega - j\nu_p & -j\,\Omega & 0 \\ j\,\Omega & \omega - j\nu_p & 0 \\ 0 & 0 & \omega - j\nu_p \end{bmatrix} \tag{3-19}$$

$$\vec{k}\,\vec{k} \equiv \begin{bmatrix} k_x k_x & k_x k_y & k_x k_z \\ k_y k_x & k_y k_y & k_y k_z \\ k_z k_x & k_z k_y & k_z k_z \end{bmatrix} \tag{3-20}$$

$\Omega = \frac{eB_o}{m}$ and $\{\quad\}^{-1}$ indicates the inverse of the matrix in brackets.

The expression for $\vec{v}_b$ is found similarly

$$\vec{v}_b = j\,\frac{e}{m}\left\{\bar{\bar{D}} - \frac{\vec{k}\,\vec{k}\,u_b^2}{\omega - \vec{V}_b\cdot\vec{k}}\right\}^{-1}\cdot\{\vec{E} + \vec{V}_b \times \mu_o\,\vec{H}\}\quad, \tag{3-21}$$

where

$$\bar{\bar{D}} \equiv \begin{bmatrix} \omega - \vec{V}_b\cdot\vec{k} - j\nu_b & -j\,\Omega & 0 \\ j\,\Omega & \omega - \vec{V}_b\cdot\vec{k} - j\nu_b & 0 \\ 0 & 0 & \omega - \vec{V}_b\cdot\vec{k} - j\nu_b \end{bmatrix} \tag{3-22}$$

If $\vec{k}\times\vec{E}$ is not zero, Eq. (3-10) can be combined with Eqs. (3-9), (3-16), (3-17), and (3-21) to yield

$$\Bigg[\vec{k}\,\vec{k} + (k_o^2 - k^2)\,\bar{\bar{I}} - k_o^2\,\frac{\omega_p^2}{\omega}\left\{\bar{\bar{C}} - \vec{k}\,\vec{k}\,\frac{u_p^2}{\omega}\right\}^{-1}$$
$$- k_o^2\,\frac{\omega_b^2}{\omega^2}\,(\omega - \vec{V}_b\cdot\vec{k})\left\{\bar{\bar{I}} + \frac{\vec{V}_b\,\vec{k}}{(\omega-\vec{V}_b\cdot\vec{k})}\right\}\cdot\left\{\bar{\bar{D}} - \frac{\vec{k}\,\vec{k}\,u_b^2}{(\omega-\vec{V}_b\cdot\vec{k})}\right\}^{-1}$$
$$\cdot\left\{\bar{\bar{I}} + \frac{\vec{k}\,\vec{V}_b}{(\omega - \vec{V}_b\cdot\vec{k})}\right\}\Bigg]\cdot\vec{E} = 0 \tag{3-23}$$

where $\bar{\bar{I}}$ is the unit matrix,

$$\vec{V}_b \vec{k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_b k_x & V_b k_y & V_b k_z \end{bmatrix} \tag{3-24}$$

and is the transpose of $\vec{k}\,\vec{V}_b$; $k_o = \omega(\mu_o\,\varepsilon_o)^{\frac{1}{2}}$.

The determinant of the matrix multiplying $\vec{E}$ in Eq. (3-23) must be zero for non-vanishing electric field and gives the dispersion relation for $\vec{k}$ as a function of ω (or ω as a function of $\vec{k}$) provided that no component of $\vec{E}$ is zero. If any component of $\vec{E}$ is zero, (e.g., $E_x = 0$) , then the dispersion relation is given by the relation that the cofactors of the corresponding column elements of the matrix (the cofactors of the xx , yx , zx terms) are zero.

We shall now consider two special wave solutions where $\vec{k} \times \vec{E} = 0$, defined to be <u>longitudinal waves</u>, and $\vec{k} \cdot \vec{E} = 0$, defined to be <u>transverse waves</u>.

3.3 <u>Longitudinal Waves-- $\vec{k} \times \vec{E} = 0$</u>

If $\vec{k} \times \vec{E} = 0$, there is no a.c. magnetic field associated with the wave, and Eq. (3-23) reduces to

$$k_o^2\left[\bar{\bar{I}} - \frac{\omega_p^2}{\omega}\left\{\bar{\bar{C}} - \vec{k}\,\vec{k}\,\frac{u_p^2}{\omega}\right\}^{-1} - \frac{\omega_b^2}{\omega}\left\{\bar{\bar{I}} + \frac{\vec{V}_b\,\vec{k}}{(\omega - \vec{V}_b\cdot\vec{k})}\right\}\right.$$
$$\left.\cdot\left\{\bar{\bar{D}} - \frac{\vec{k}\,\vec{k}\,u_b^2}{(\omega - \vec{V}_b\cdot\vec{k})}\right\}^{-1}\right]\cdot\vec{E} = 0 \quad . \tag{3-25}$$

We now consider a particular example of a longitudinal wave where $\vec{k}$ is directed along the positive z axis, the direction of the d.c. magnetic field and the beam velocity. Equation (3-25) is then written

$$k_o^2\left[\bar{\bar{I}} - \frac{\omega_p^2}{\omega}\left\{\bar{\bar{C}} - \vec{k}_z\,\vec{k}_z\,\frac{u_p^2}{\omega}\right\}^{-1} - \frac{\omega_b^2}{\omega}\left\{\bar{\bar{I}} + \frac{\vec{V}_b\,\vec{k}_z}{(\omega - V_b\,k_z)}\right\}\right.$$
$$\left.\cdot\left\{\bar{\bar{D}} - \frac{\vec{k}_z\,\vec{k}_z\,u_b^2}{(\omega - V_b\,k_z)}\right\}^{-1}\right]\cdot\begin{Bmatrix} 0 \\ 0 \\ E_z \end{Bmatrix} = 0 \quad . \tag{3-26}$$

The dispersion relation is given by the condition that the third column of the matrix multiplying $\vec{E}$ be zero, that is, the xz , yz , and zz elements are zero.

With $\vec{k}$ directed along the positive z direction, the matrix

$$\left\{\bar{\bar{C}} - \vec{k}_z\,\vec{k}_z\,\frac{u_p^2}{\omega}\right\}^{-1} = \frac{1}{\Omega_p^2 - \Omega^2}\begin{bmatrix} \Omega_p & j\Omega & 0 \\ -j\Omega & \Omega_p & 0 \\ 0 & 0 & \dfrac{\Omega_p^2 - \Omega^2}{\Omega_p - \dfrac{k_z^2\,u_p^2}{\omega}} \end{bmatrix} \tag{3-27}$$

where

$$\Omega_p \equiv \omega - j\nu_p \quad . \tag{3-28}$$

Similarly,

$$\left\{ \overline{\overline{D}} - \frac{\vec{k}_z \vec{k}_z u_b^2}{(\omega - V_b k_z)} \right\}^{-1} =$$

$$\frac{1}{\Omega_b^2 - \Omega^2} \begin{bmatrix} \Omega_b & j\Omega & 0 \\ -j\Omega & \Omega_b & 0 \\ 0 & 0 & \dfrac{\Omega_b^2 - \Omega^2}{\Omega_b - \dfrac{k_z^2 u_b^2}{(\omega - V_b k_z)}} \end{bmatrix} \tag{3-29}$$

where

$$\Omega_b \equiv \omega - V_b k_z - j\nu_b \tag{3-30}$$

and

$$\left\{ \overline{\overline{I}} + \frac{\vec{V}_b \vec{k}_z}{(\omega - V_b k_z)} \right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\omega}{\omega - V_b k_z} \end{bmatrix} \tag{3-31}$$

Since the xz , and yz terms of the matrix multiplying $\vec{E}_z$ are identically zero, the only relation obtained from the dispersion equation is that the zz term must vanish. Thus, the dispersion relation becomes:

$$1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p) - k_z^2 u_p^2} - \frac{\omega_b^2}{(\omega - k_z V_b)(\omega - k_z V_b - j\nu_b) - k_z^2 u_b^2} = 0 \quad . \qquad (3\text{-}32)$$

Calculations were carried out for the longitudinal waves given by the dispersion equation (3-32). In particular, the longitudinal waves which indicate the possibility of amplification, that is, have both positive real and imaginary propagation constants for real excitation frequency, were investigated by obtaining computer solutions to the fourth order dispersion equation. The results for various parameter values are shown in Figs. (3-1)-(3-20). The results of these computations can be summarized in the following way:

<u>Warm-Beam--Warm-Plasma</u> (No Collisions—$\nu_p = \nu_b = 0$)

1. For values of beam velocity, V_b , much greater than the adiabatic sound speed in the plasma, u_p , the warm plasma calculations differ from the cold beam-plasma case only close to the region $\omega = \omega_p$, where the effect of plasma temperature limits the real and imaginary parts of the propagation constants to finite values (Figs. (3-1) and (3-2)).

2. It was found that maximum gain is obtained when $\omega \simeq \omega_p$ and the beam velocity is greater than but on the order of the adiabatic sound speed in the plasma (Fig. (3-3)).

3. The effect of the maximum gain parameter $(V_b/u_p)_{max}$ was to peak the gain and make the real part of the propagation constant a

smooth function of ω/ω_p so that no sharp break in the real part of k_z versus ω/ω_p occurred. Any value of V_b/u_p not corresponding to the maximum gain value left a hump in the real part of the k_z versus ω curve at $\omega \approx \omega_p$ (Figs. (3-3), (3-4), and (3-5)).

4. It was found that the maximum gain obtained near the plasma frequency for large values of V_b/u_p saturated for values of beam density such that $(\omega_b/\omega_p)^2 \gtrsim 4$ (Fig. (3-6)).

5. The effects of beam temperatures are not seen unless $u_b \sim u_p$ in which case the beam temperatures tend to decrease the amplifier gain and increase the range of ω over which gain occurs.

6. Figures (3-7) and (3-8), show all four solutions to the dispersion relation of Eq. (3-32) when collisions are absent. All solutions with complex parts will exist with complex conjugate counterparts since this dispersion equation without collisions is real.

Warm-Beam--Warm-Plasma (Collisions Included)

1. It is shown in Figs. (3-9)-(3-20) that collisional effects are most noticeable when:

(a) the collision frequencies are on the order of a tenth or more of their respective plasma frequencies

(b) the beam velocity, V_b, is much greater than the sound speed of the plasma, u_p

(c) the beam density is small compared to the plasma density.

UNBOUNDED BEAM-PLASMA CALCULATIONS

For the graphical results shown in Figures (3-1)-(3-20) the following labels are used:

$$A = \left(\frac{\omega}{\omega_p}\right)^2$$

$$B = \left(\frac{\omega_b}{\omega_p}\right)^2$$

$$C = \frac{V_o}{u_p}$$

$$D = \left(\frac{u_b}{u_p}\right)^2$$

$$E = \frac{\nu_p}{\omega_p}$$

$$F = \frac{\nu_b}{\omega_b} \quad .$$

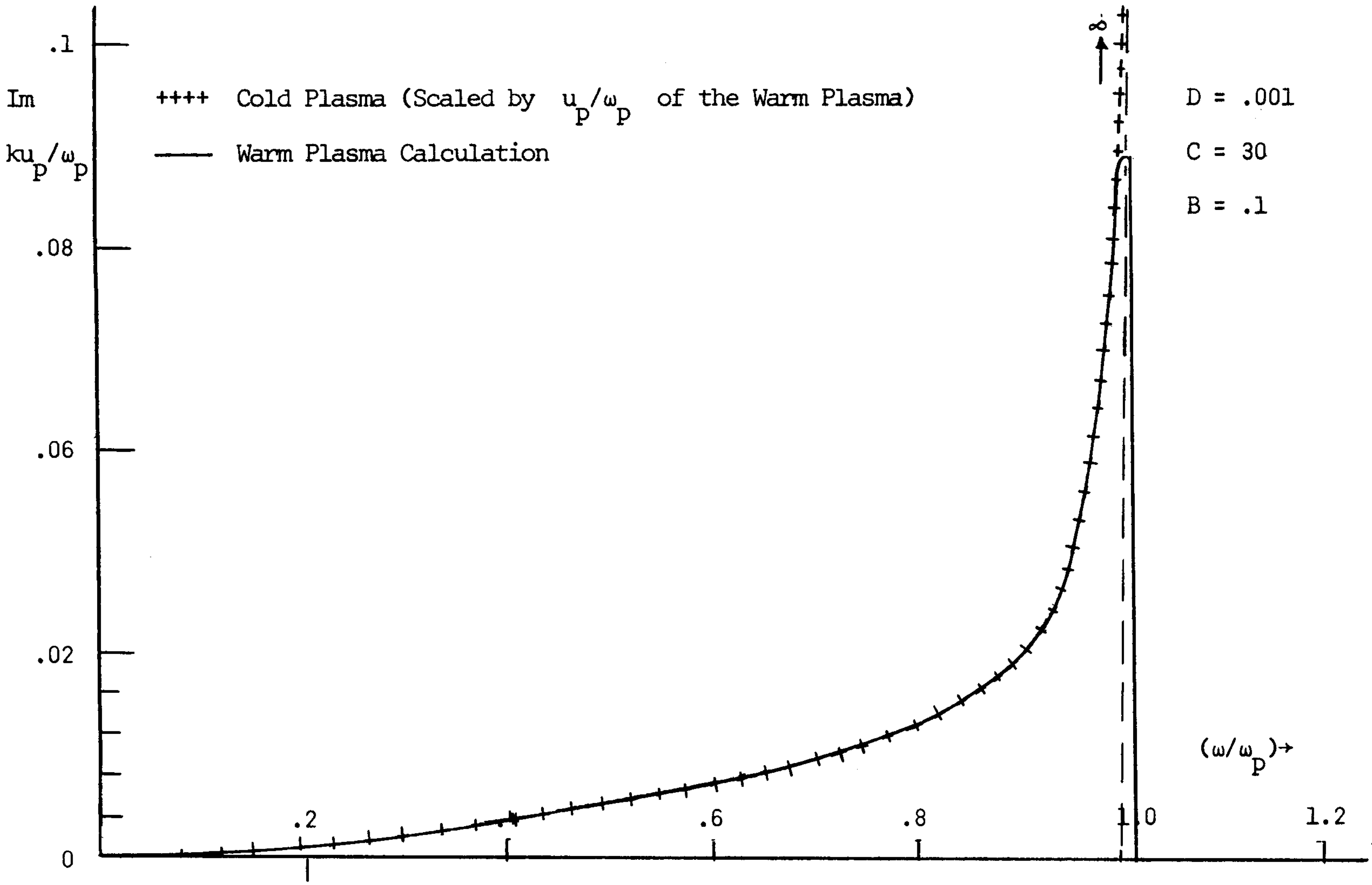

Fig. (3-1) IMAGINARY PART OF $k(u_p/\omega_p)$ VERSUS (ω/ω_p)

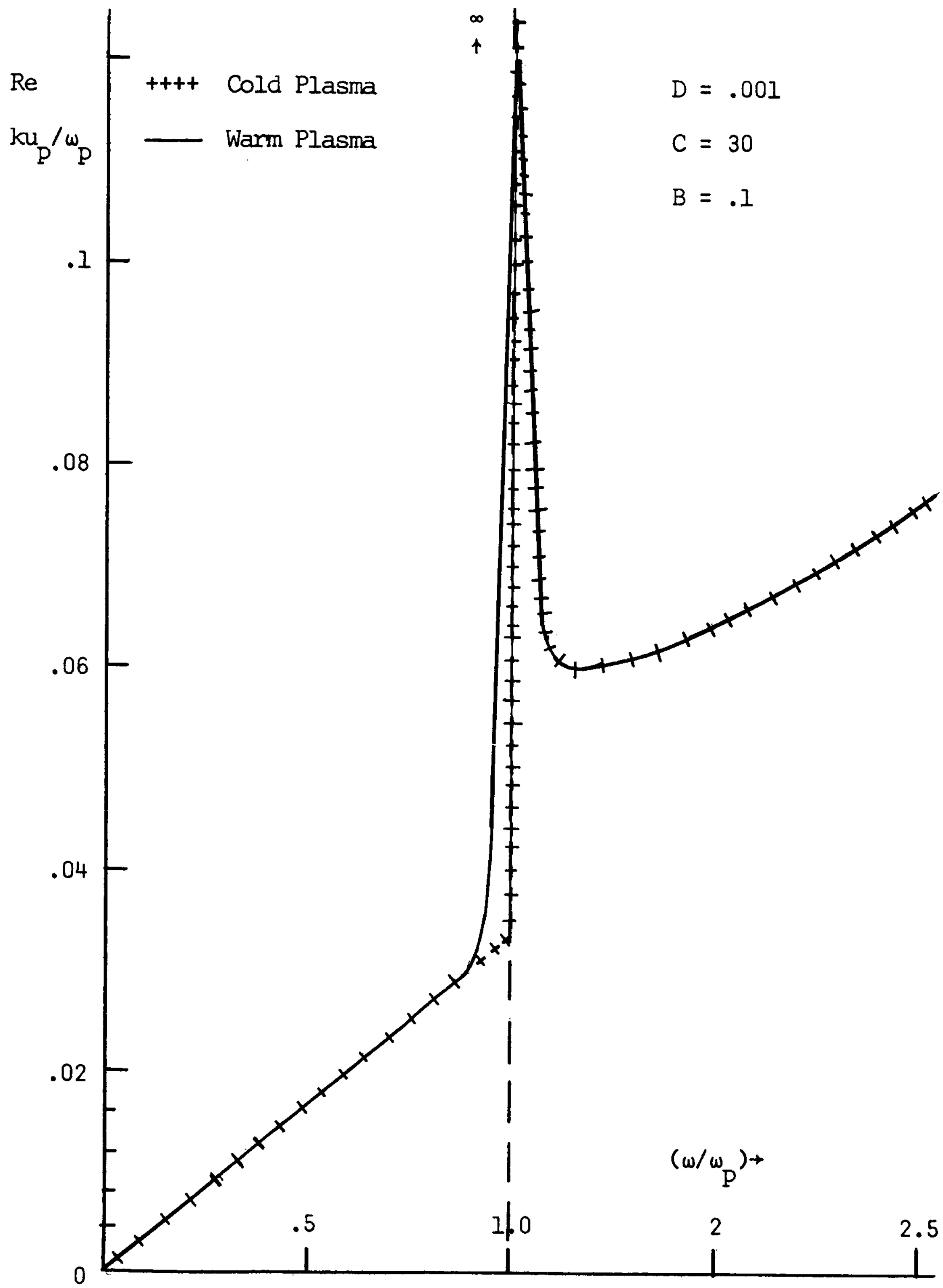

Fig. (3-2) REAL PART OF $k(u_p/\omega_p)$ VERSUS (ω/ω_p)

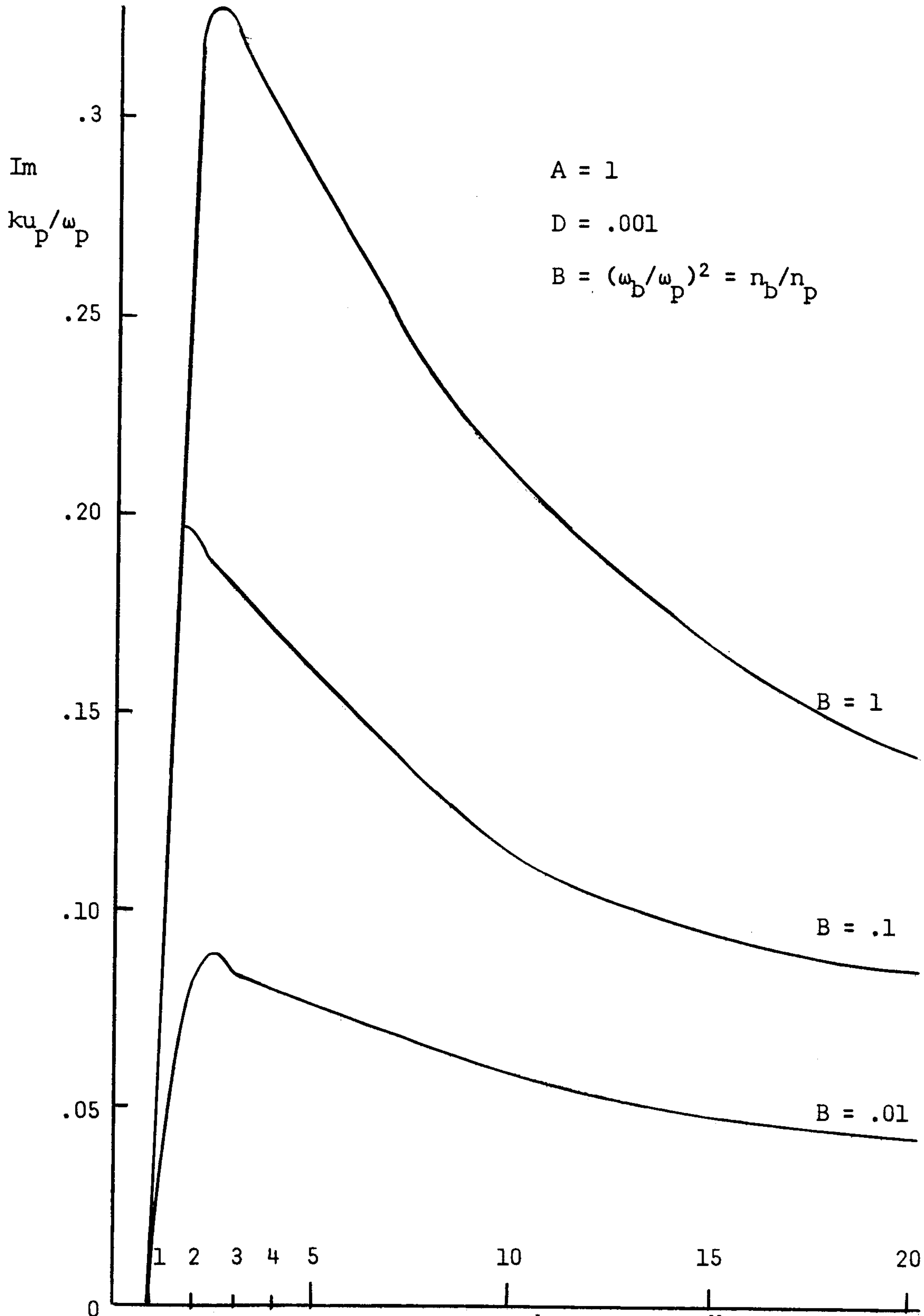

Fig. (3-3) IMAGINARY PART OF $\frac{ku_p}{\omega_p}$ VERSUS $\frac{V_o}{u_p}$

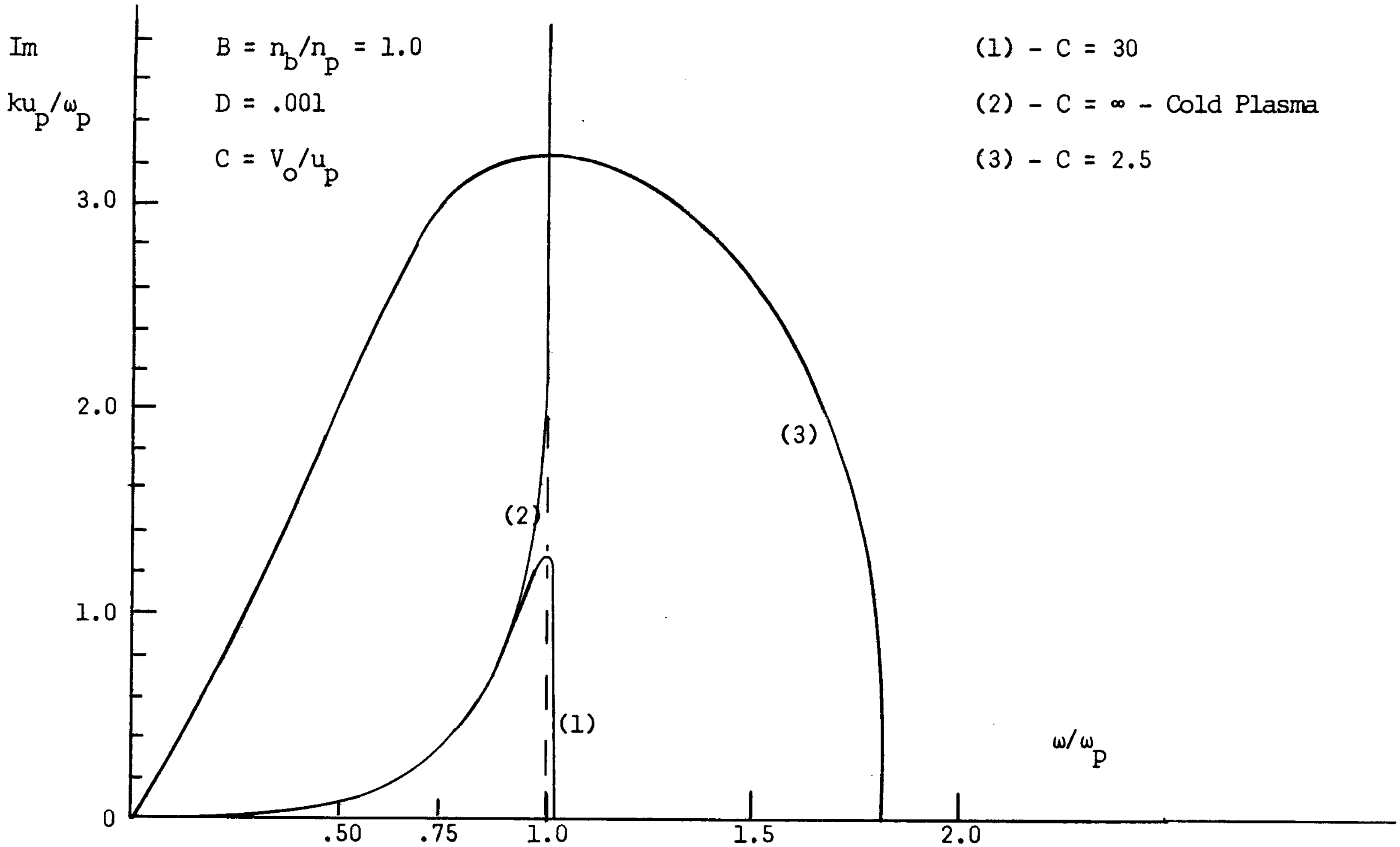

Fig. (3-4) IMAGINARY PART OF $k(u_p/\omega_p)$ VERSUS ω/ω_p

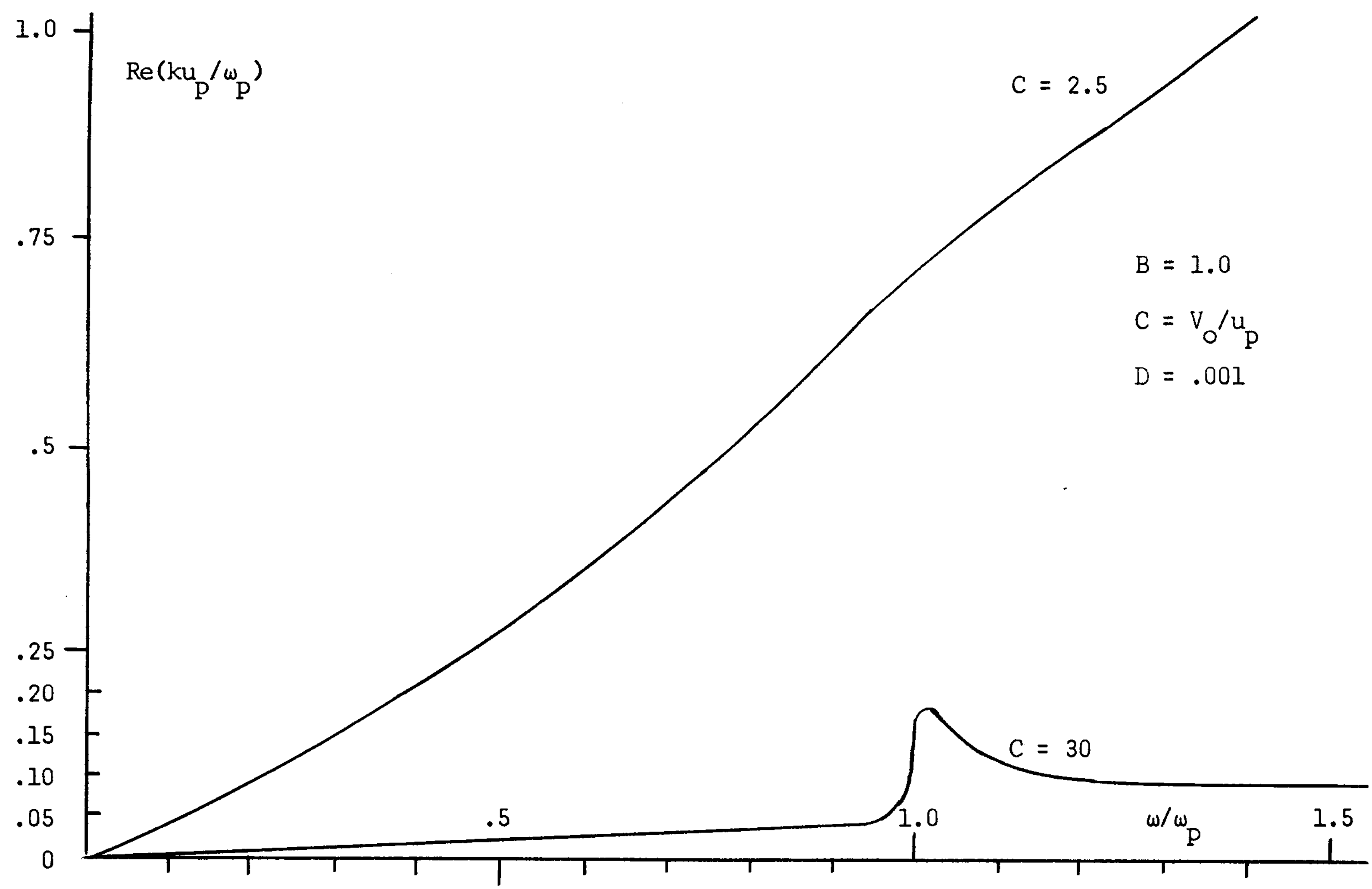

Fig. (3-5) REAL PART OF $k(u_p/\omega_p)$ VERSUS (ω/ω_p)

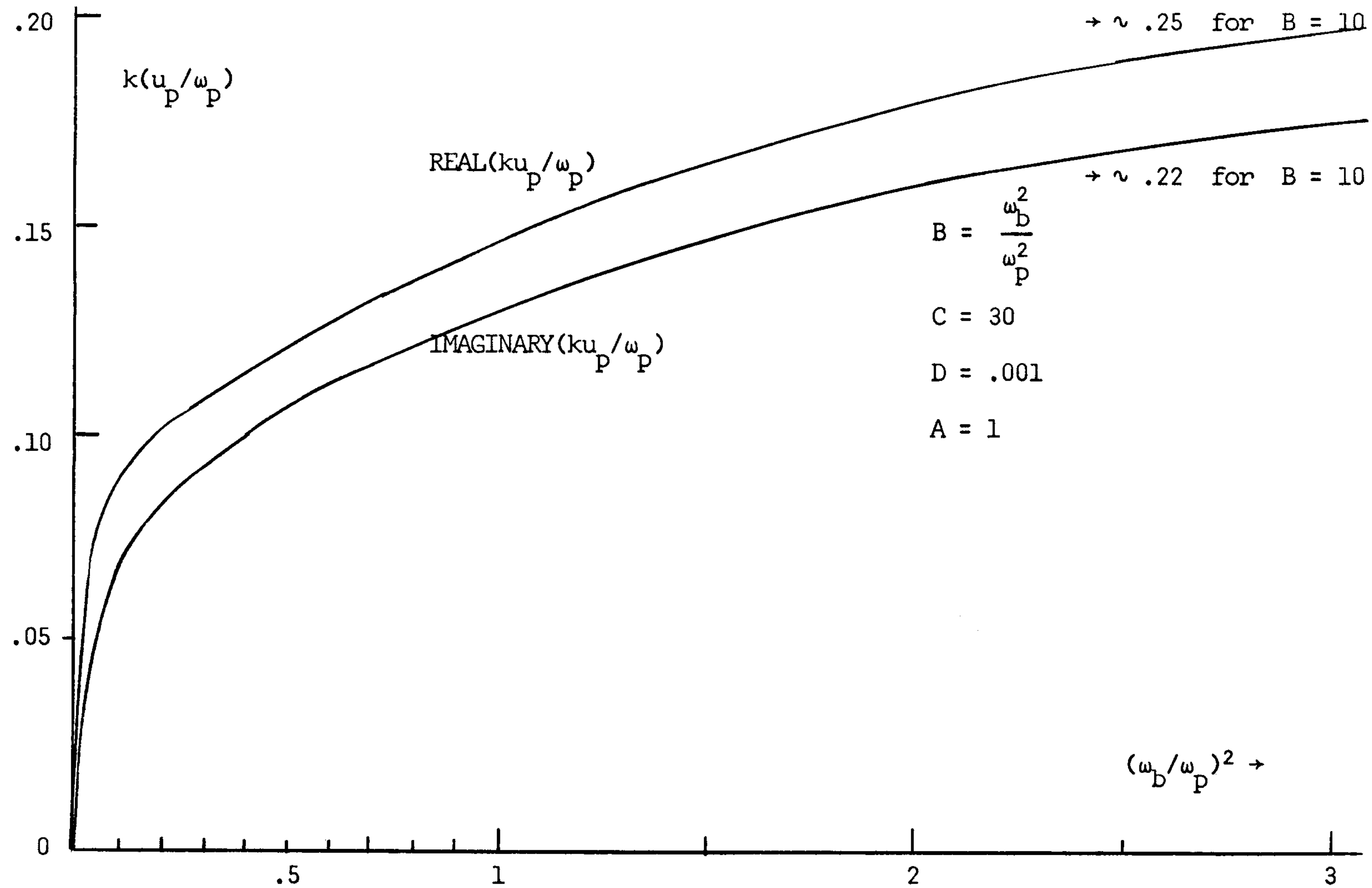

Fig. (3-6) $k(u_p/\omega_p)$ VERSUS $(\omega_b/\omega_p)^2$

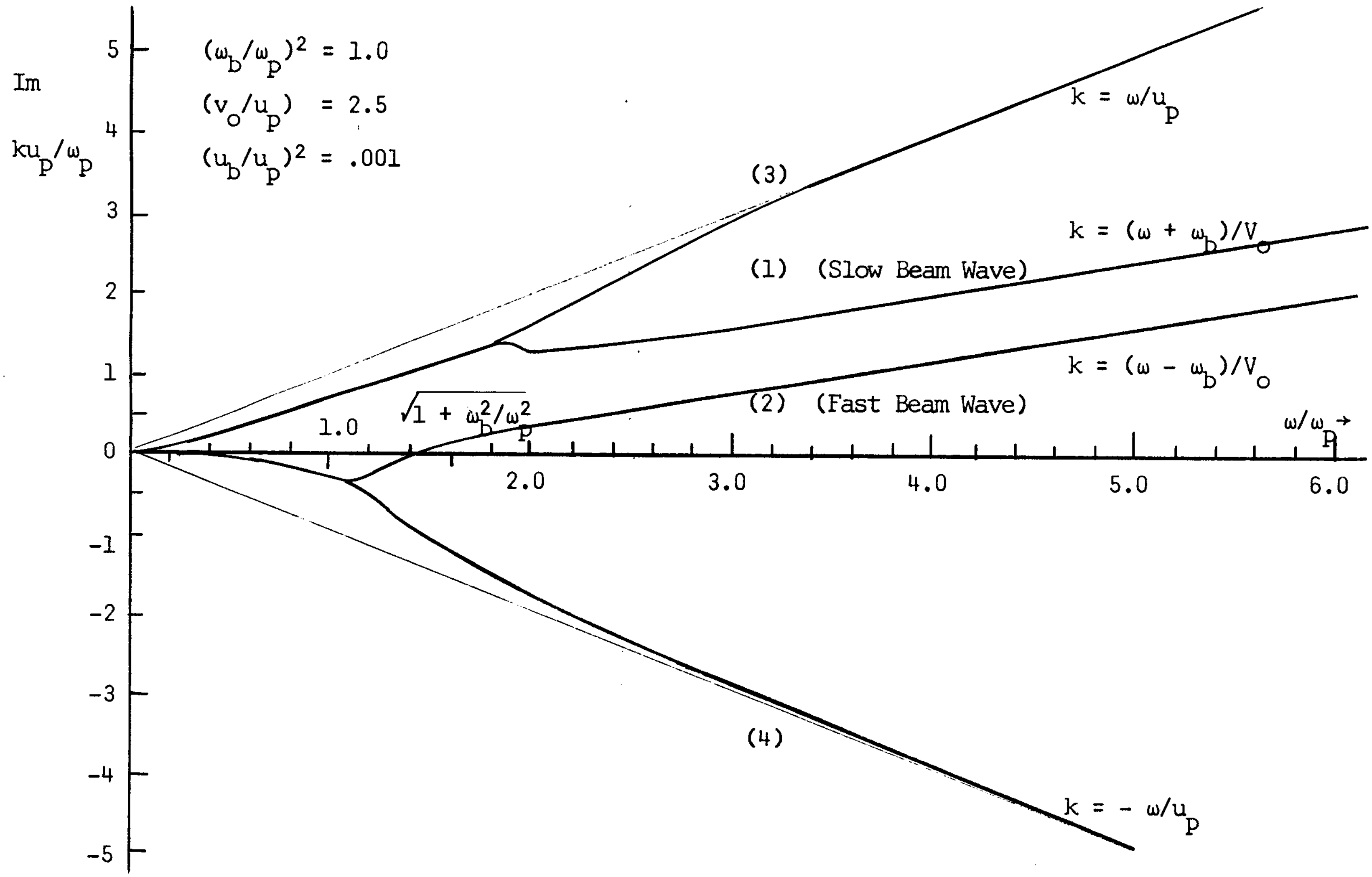

Fig. (3-7) REAL PART OF $k(u_p/\omega_p)$ VERSUS (ω/ω_p)

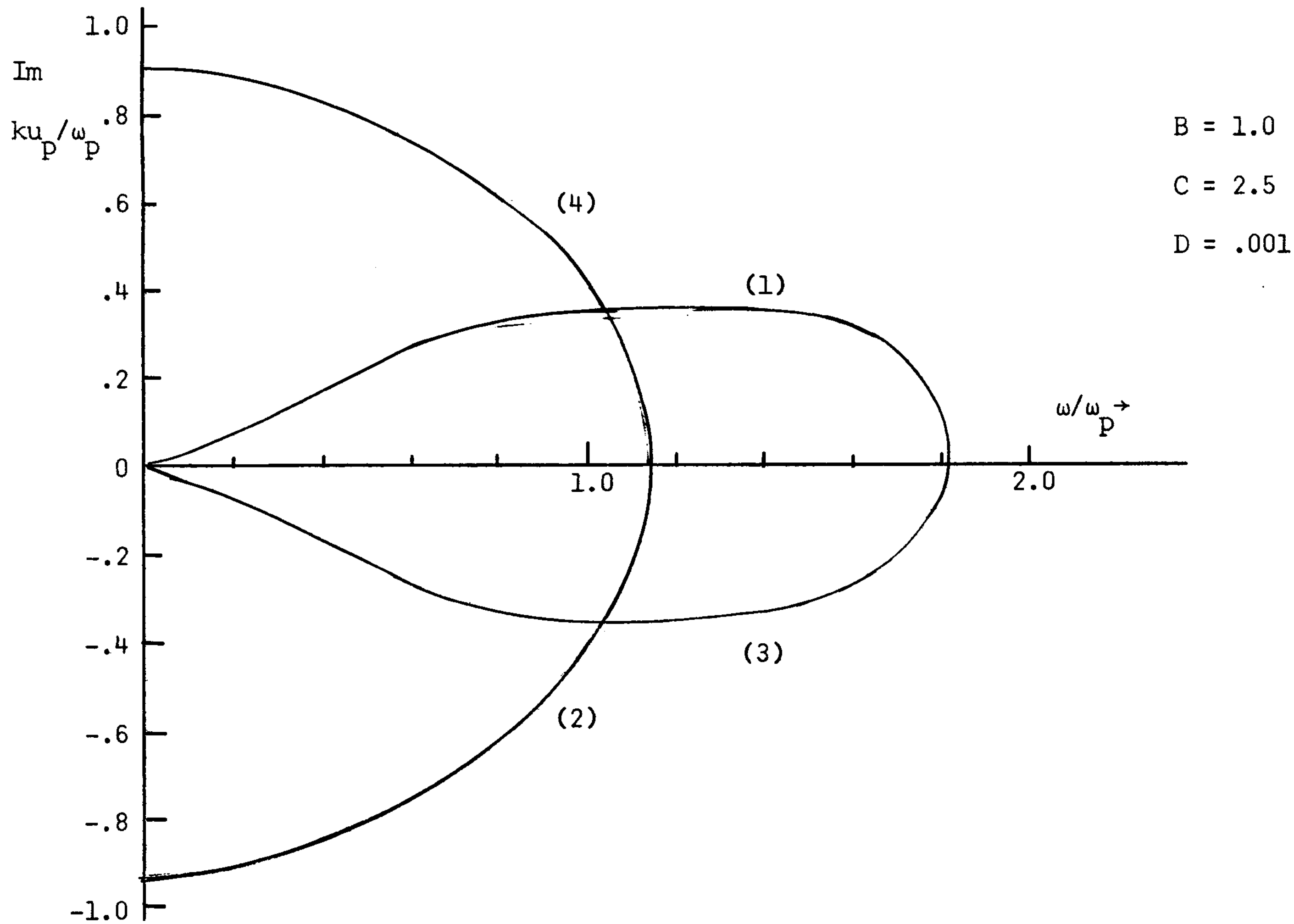

Fig. (3-8) IMAGINARY PART OF $k(u_p/\omega_p)$ VERSUS (ω/ω_p)

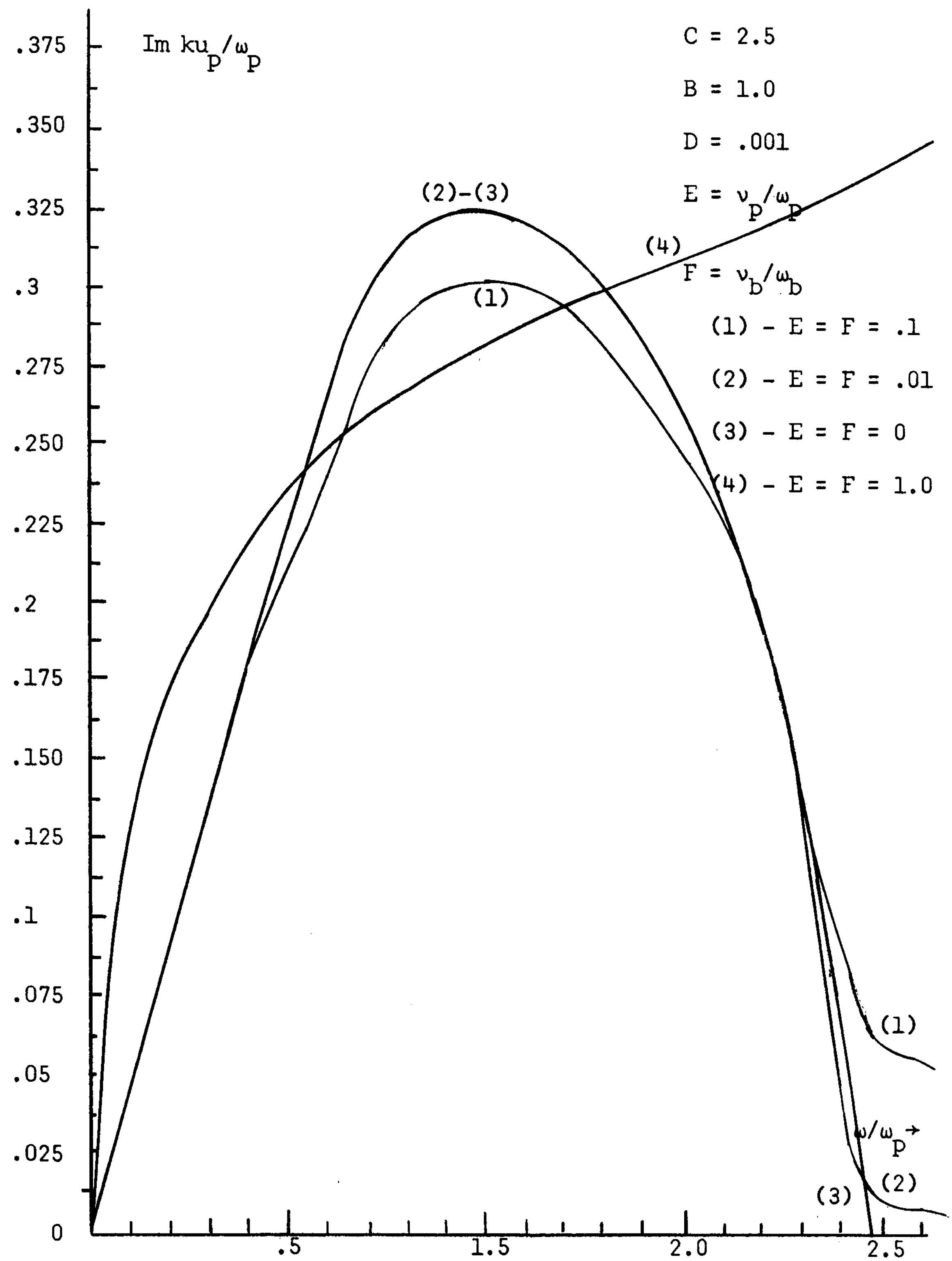

Fig. (3-9) $\mathrm{Im}(ku_p/\omega_p)$ VERSUS ω/ω_p

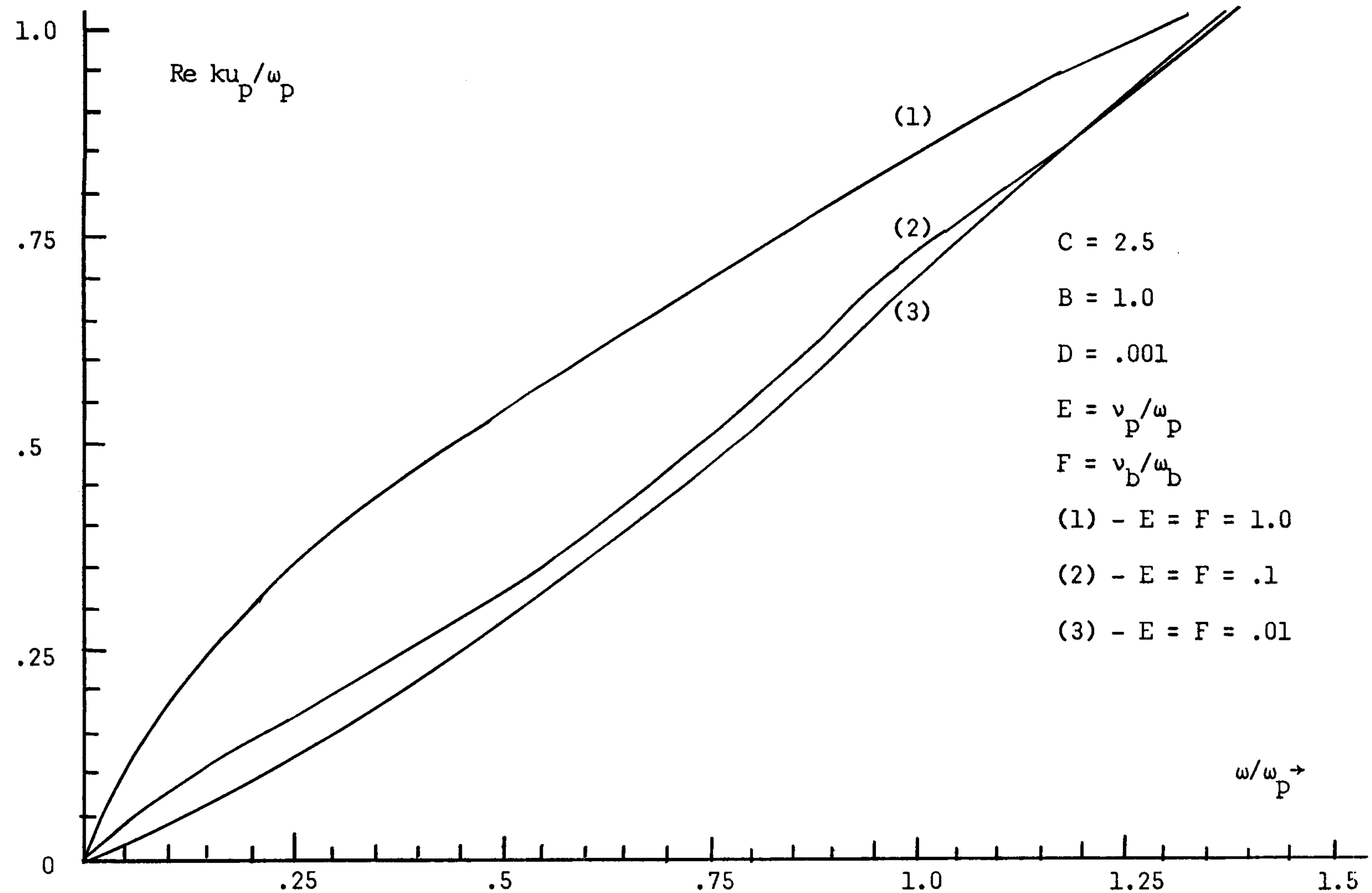

Fig. (3-10) $Re(ku_p/\omega_p)$ VERSUS ω/ω_p

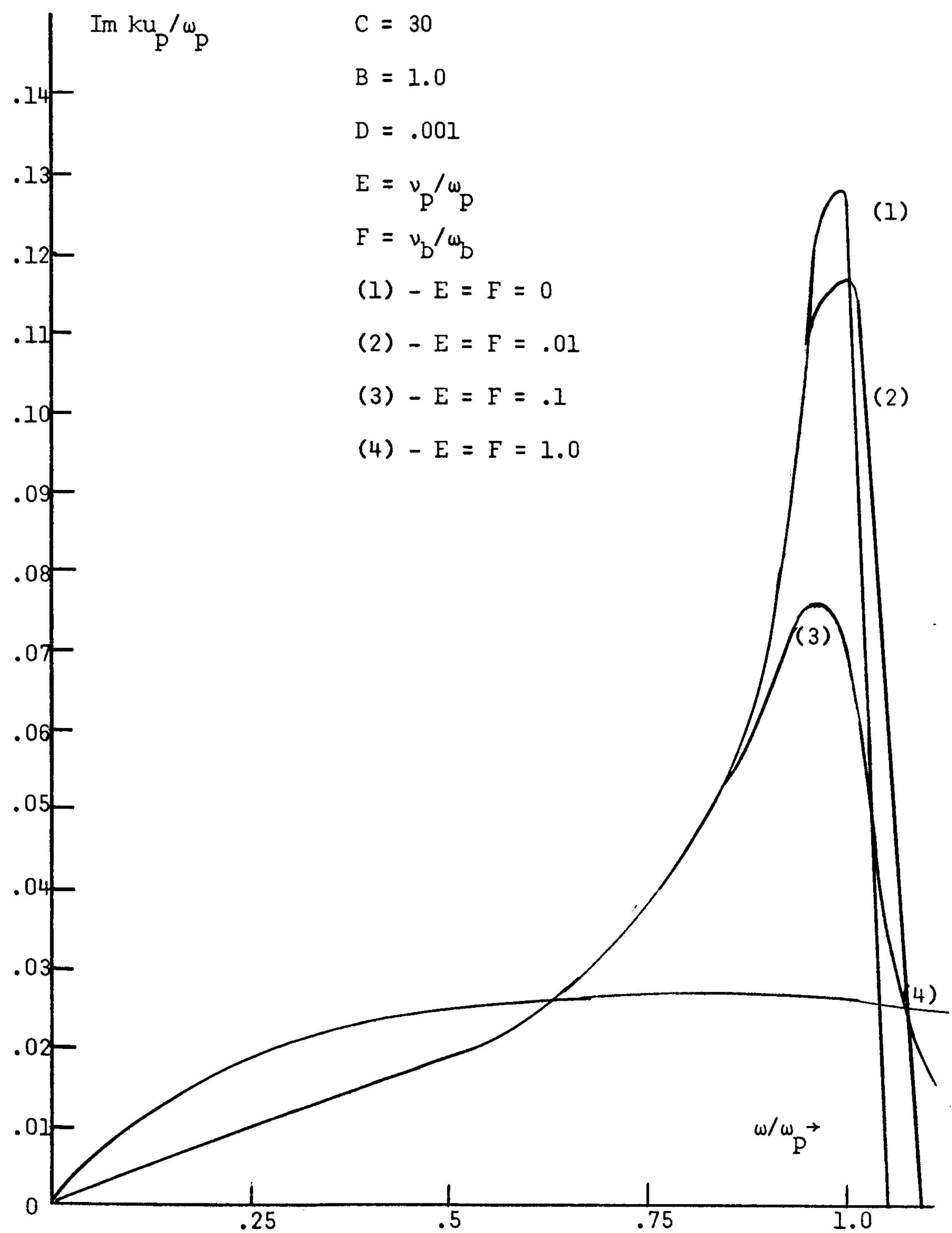

Fig. (3-11) Im(ku_p/ω_p) VERSUS ω/ω_p

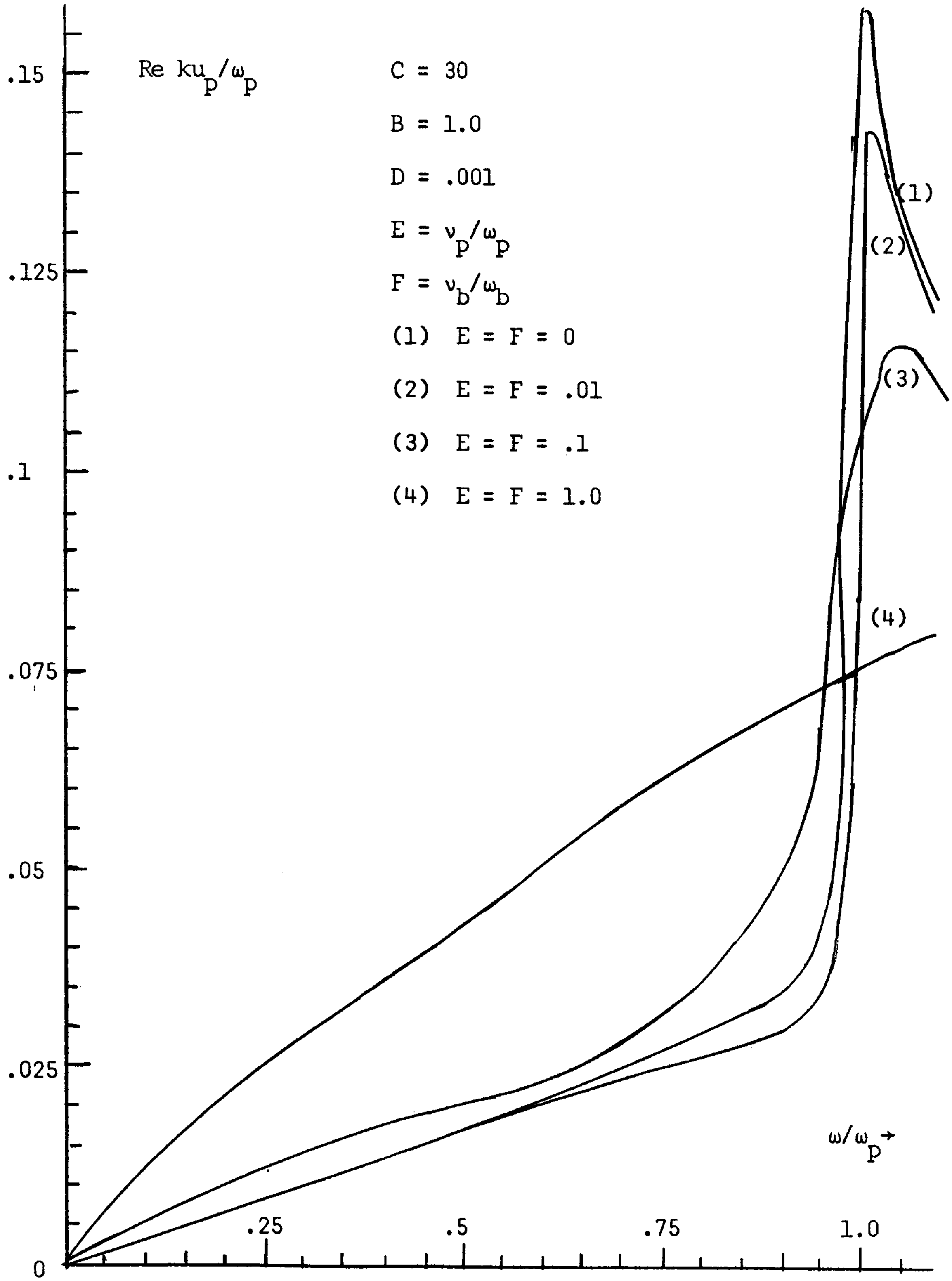

Fig. (3-12) Re(ku_p/ω_p) VERSUS ω/ω_p

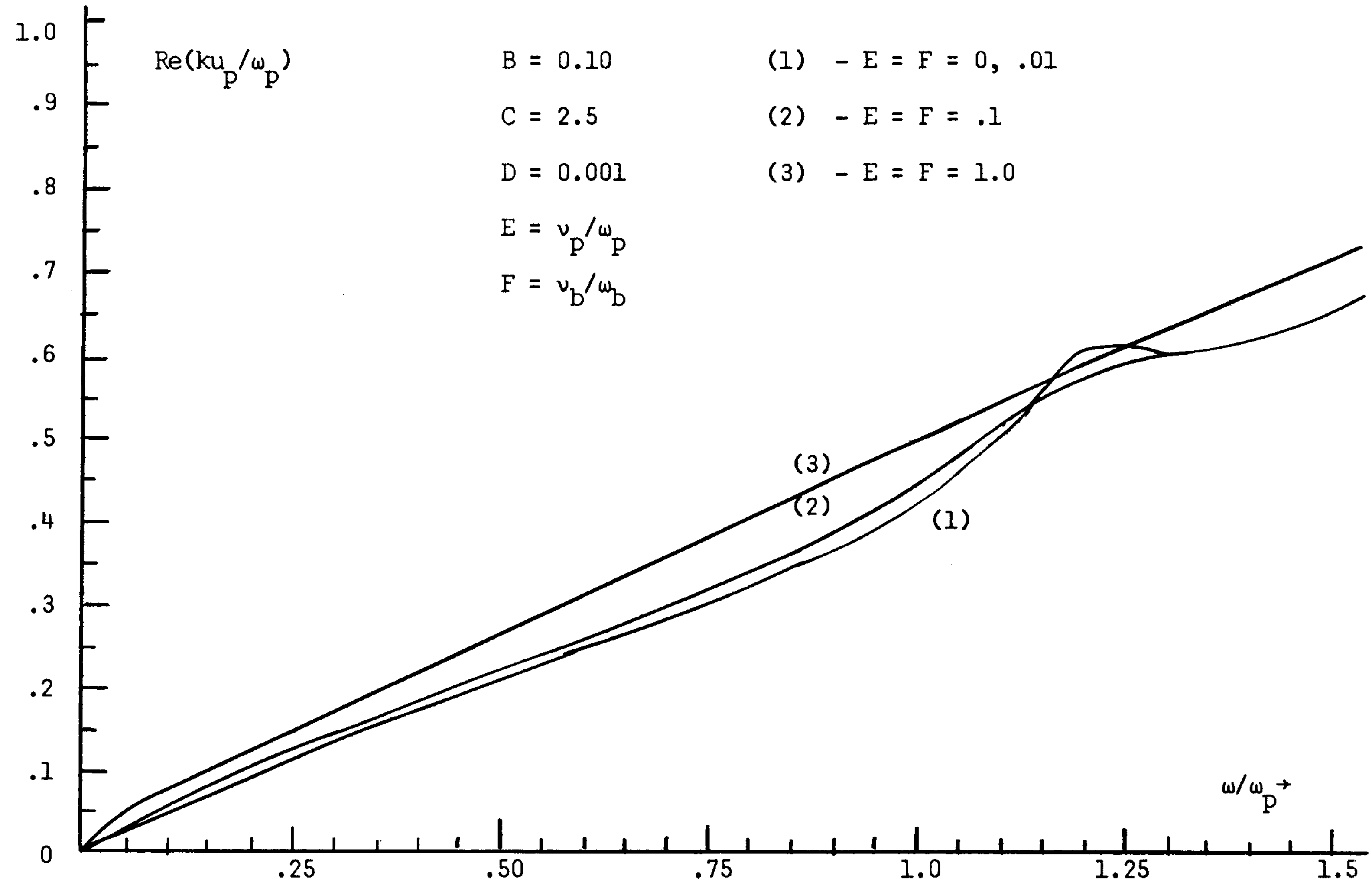

Fig. (3-13) REAL PART OF ku_p/ω_p VERSUS ω/ω_p

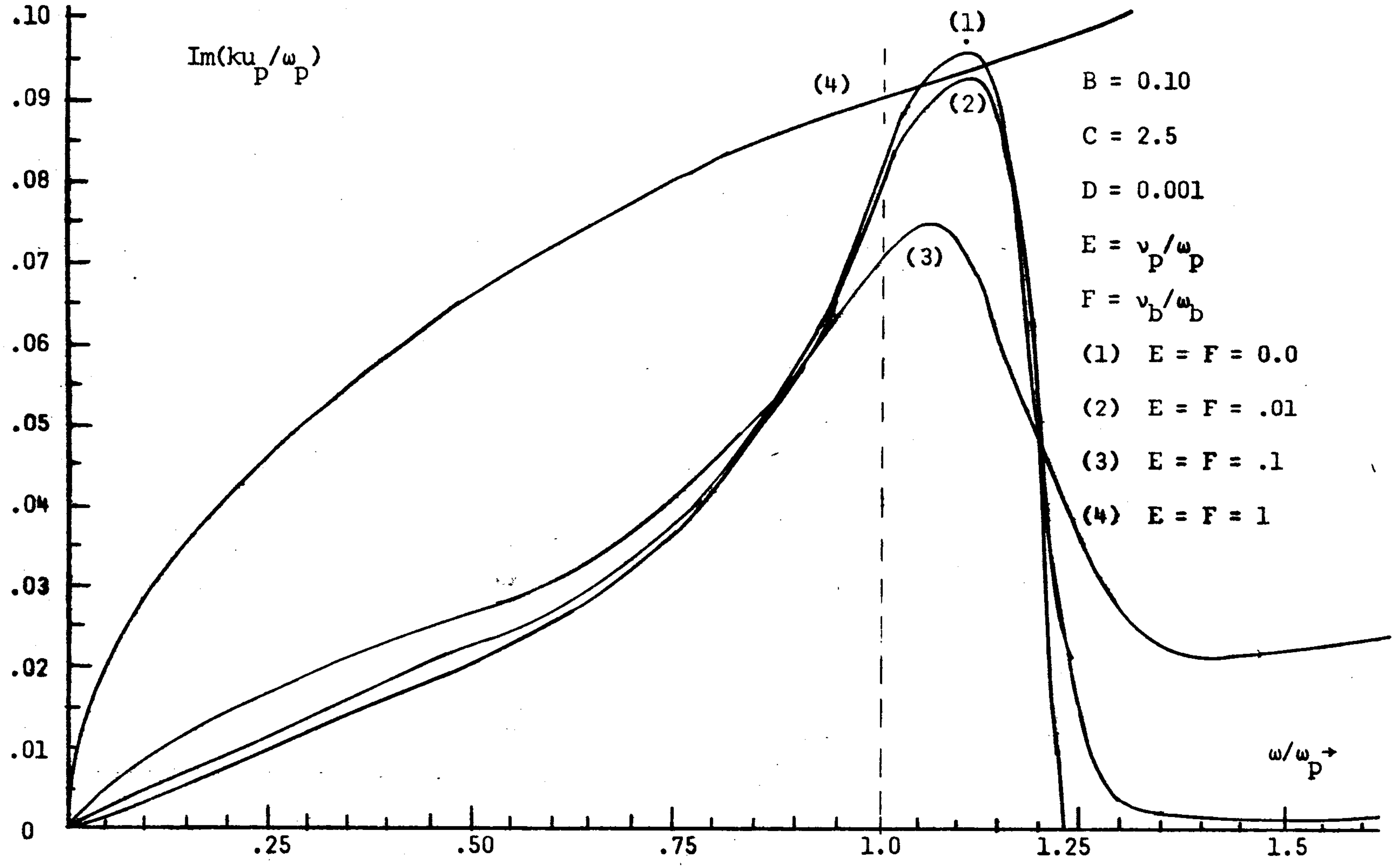

Fig. (3-14) IMAGNIARY PART OF ku_p/ω_p VERSUS ω/ω_p

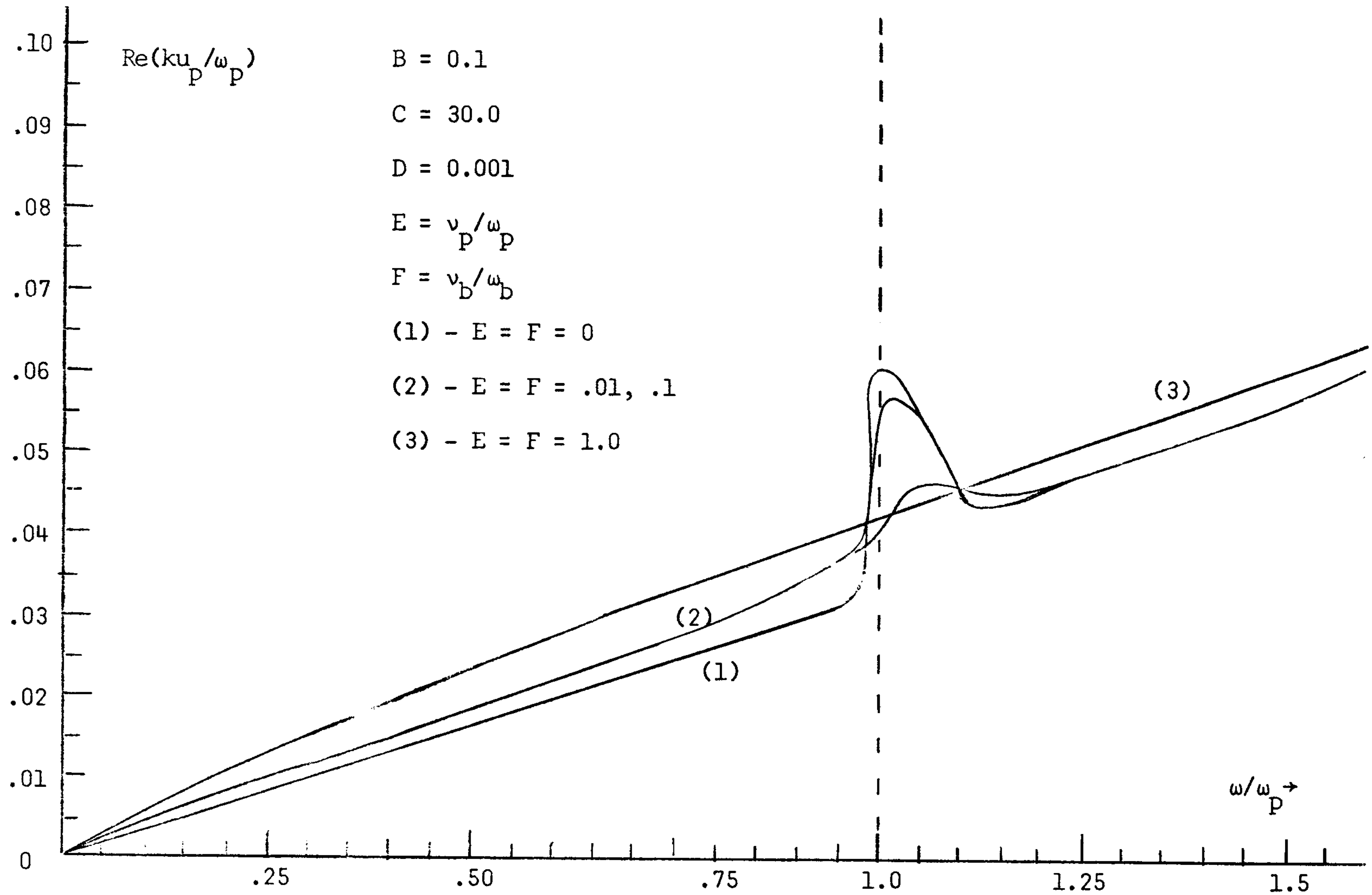

Fig. (3-15) REAL PART OF (ku_p/ω_p) VERSUS ω/ω_p

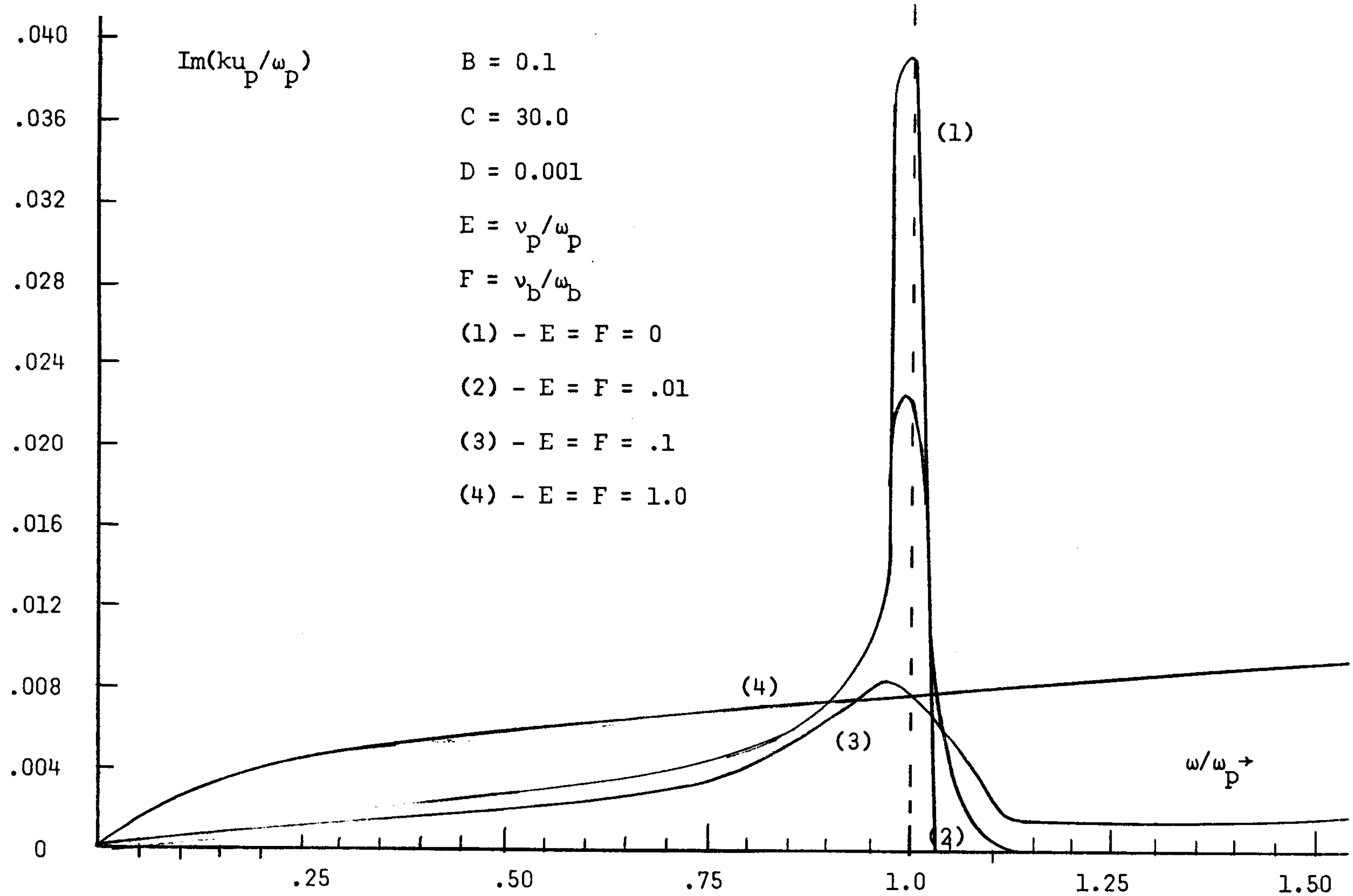

Fig. (3-16) IMAGINARY PART OF ku_p/ω_p VERSUS ω/ω_p

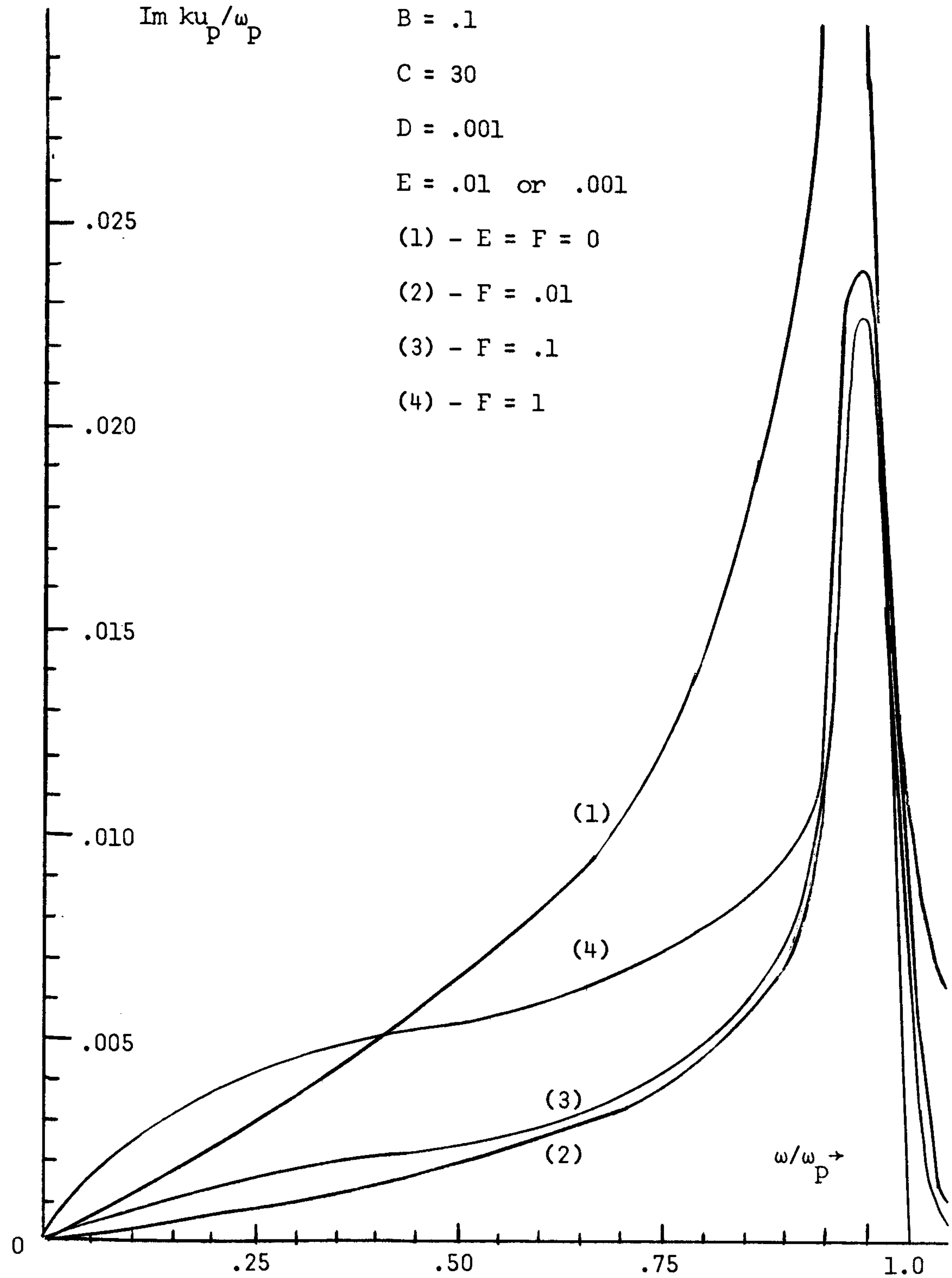

Fig. (3-17) IMAGINARY PART OF ku_p/ω_p VERSUS ω/ω_p

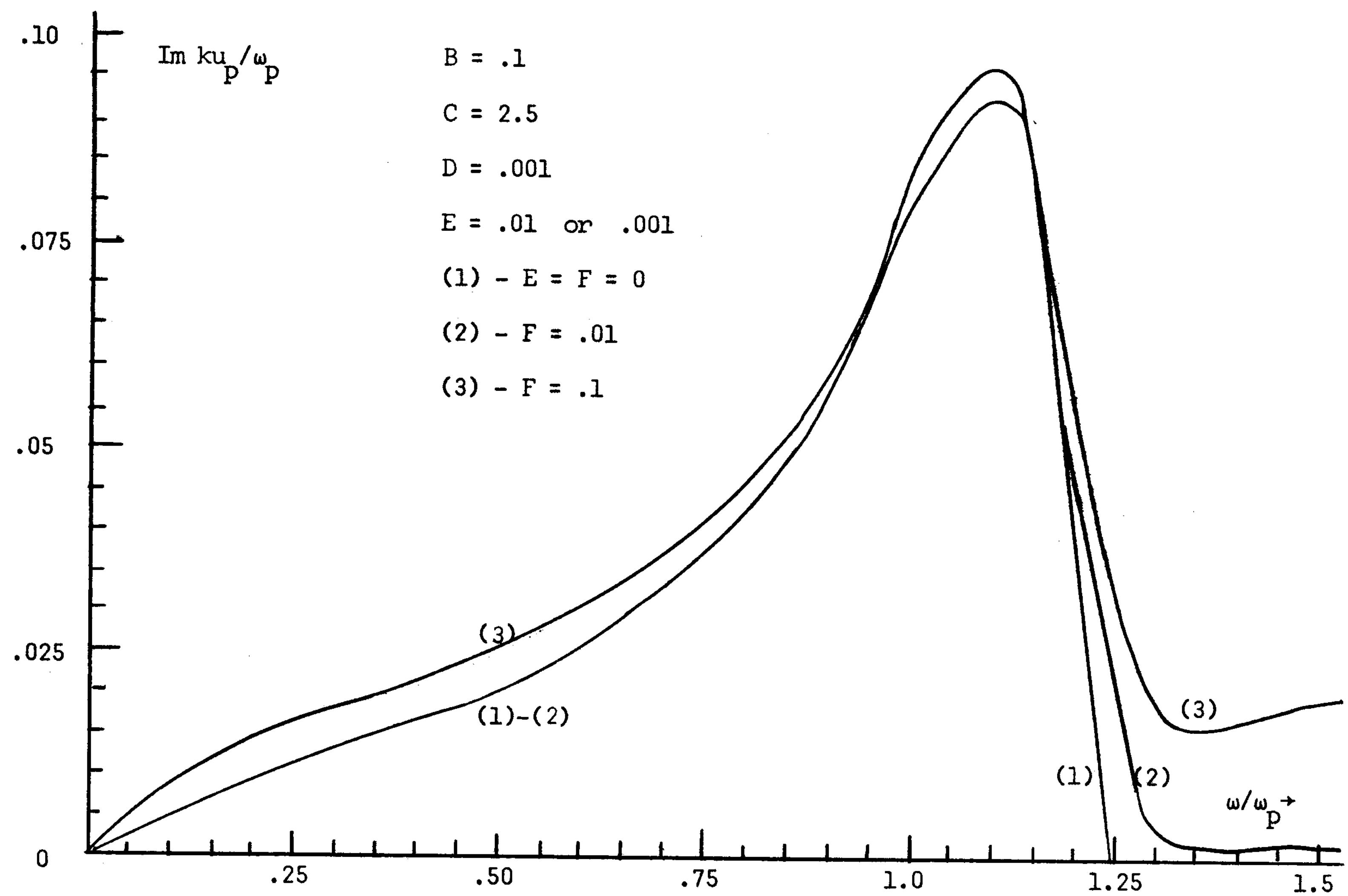

Fig. (3-18) IMAGINARY PART OF ku_p/ω_p VERSUS ω/ω_p

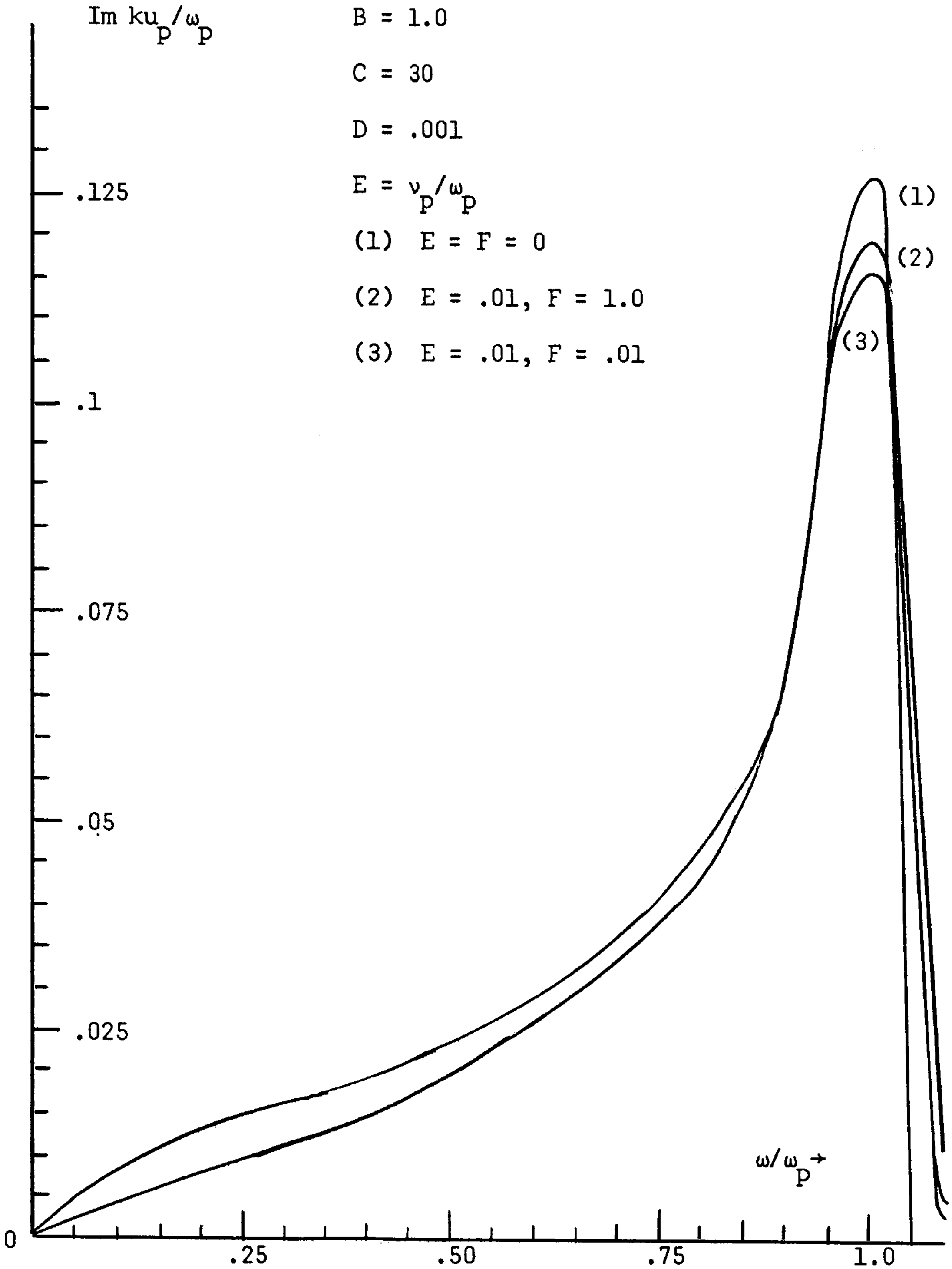

Fig. (3-19) IMAGINARY PART OF ku_p/ω_p VERSUS ω/ω_p

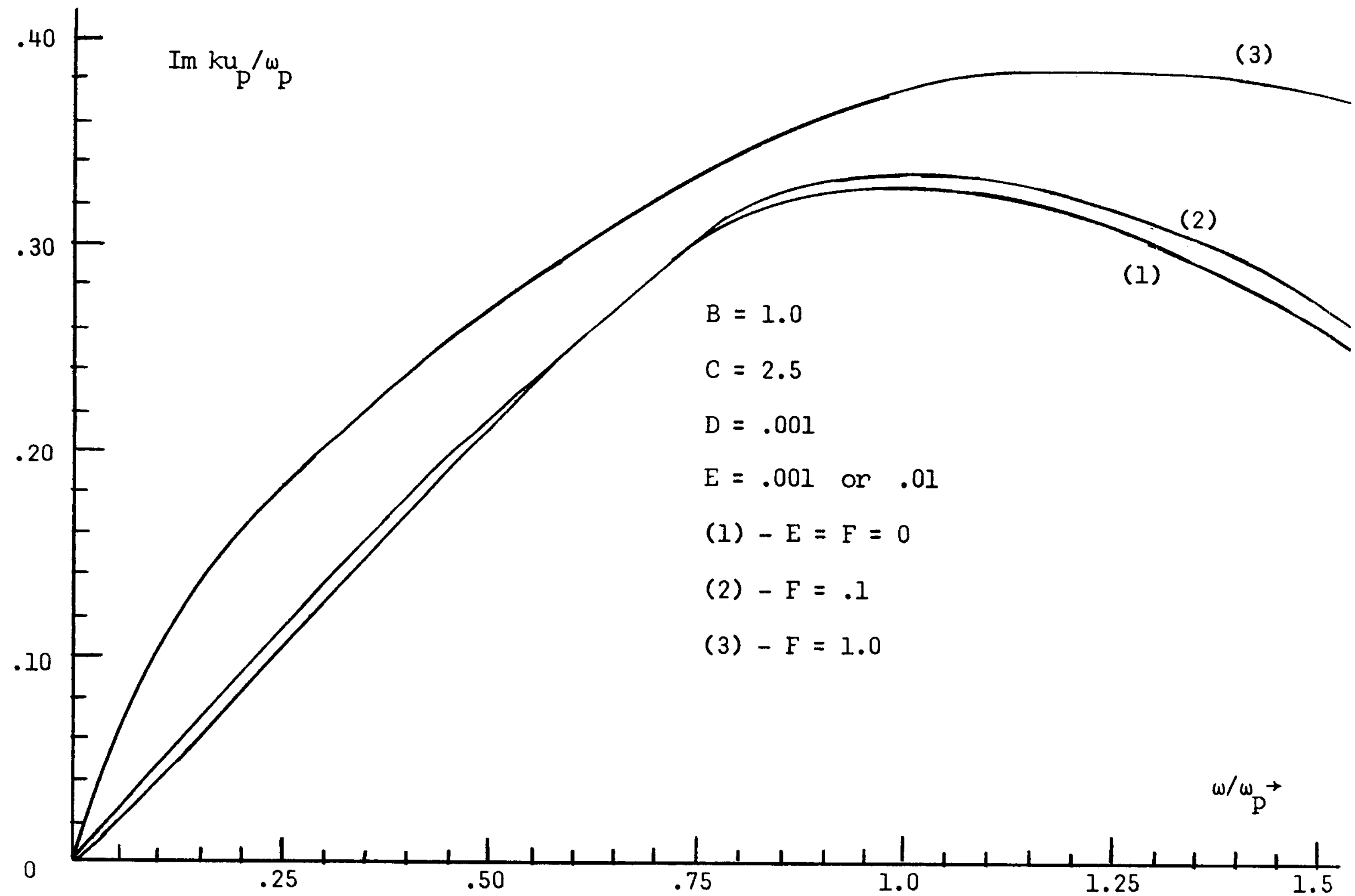

Fig. (3-20) IMAGINARY PART OF ku_p/ω_p VERSUS ω/ω_p

2. In Figures (3-17)-(3-20), it is shown that the collision frequency of the beam electrons tends to increase the gain of the amplifier while collisions in the plasma decrease gain. The reason for this is shown by the fact that the a.c. velocity of the beam is in phase with the electric field when ω_p is greater than ω. After collision, the velocity becomes 180° out of phase and can give energy to the field.

3. It is somewhat questionable whether collision frequencies on the order of the plasma frequency could be obtained in practice for a beam-plasma system, however, the calculations were carried out to indicate the general mathematical trends and effects of collisions.

Physical Mechanism for Longitudinal Wave Amplification. We now investigate the physical mechanism by which longitudinal waves are amplified. To do this, we shall consider the simplest case of an infinite system with no collisions or temperatures and with a.c. velocities directed along the beam and the d.c. magnetic field. For longitudinal waves, the curl of the electric field vanishes, $(\vec{k} \times \vec{E}) = 0$, which makes the a.c. magnetic field also vanish. The dispersion relation, then, is derived from (3-10) which is just the condition that the sum of the displacement, plasma, and beam currents be zero.

Upon substitution of the other constitutive relations, (3-11)-(3-14), the Eq. (3-10) becomes:

$$j\omega\, \epsilon_o\, \vec{E}_z\left(1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega(\omega - k_z V_b)} - \frac{\omega_b^2\, k_z\, V_b}{\omega(\omega - k_z V_b)^2}\right) = 0 \tag{3-33}$$

where the terms correspond to displacement, plasma, beam, and beam-space-charge drift currents respectively. The dispersion relation, then, is

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{(\omega - k_z\, V_b)^2} = 0 \tag{3-34}$$

which is the result derived in (3-32) for a cold, collisionless system. Equation (3-34) can be solved for k_z giving:

$$(\omega - k_z\, V_b)^2 = \frac{\omega_b^2\, \omega^2}{\omega^2 - \omega_p^2} \tag{3-35}$$

$$(\omega - k_z\, V_b) = \pm \frac{\omega_b\, \omega}{(\omega^2 - \omega_p^2)^{\frac{1}{2}}} = \pm\, j\, \frac{\omega_b\, \omega}{(\omega_p^2 - \omega^2)^{\frac{1}{2}}} \tag{3-36}$$

$$k_z = \frac{\omega}{V_b}\left(1 \mp j\, \frac{\omega_b}{(\omega_p^2 - \omega^2)^{\frac{1}{2}}}\right) \tag{3-37}$$

Equation (3-37) indicates that the system could possibly amplify as long as the plasma frequency is greater than the excitation frequency, ω, since k_z can have both positive real and imaginary parts. Furthermore, the equation suggests that in the absence of collisions and temperatures, the gain could become infinite when the system is excited at the plasma frequency.

Introducing only beam collisions or beam temperature does not change the possibility of infinite gain, whereas plasma collisions and temperature restrict the gain to a finite value.

We can see the dependence of each of the current terms on frequency if Eqs. (3-35)-(3-37) are substituted into Eq. (3-33). The result is:

$$j\omega\,\varepsilon_o\,\vec{E}_z\left\{1-\frac{\omega_p^2}{\omega^2}\pm j\,\frac{\omega_b(\omega_p^2-\omega^2)^{\frac{1}{2}}}{\omega^2}\right.$$

$$\left.-\left(1-\frac{\omega_p^2}{\omega^2}\pm j\,\frac{\omega_b(\omega_p^2-\omega^2)^{\frac{1}{2}}}{\omega^2}\right)\right\}=0\quad, \tag{3-38}$$

where, again, each term corresponds to displacement, plasma, beam, and beam-space-charge drift currents respectively. A phasor diagram of these currents is shown in Fig. (3-21) corresponding to the case where $\mathrm{Im}(k_z)$ is positive (bottom signs) and ω_p is greater than ω.

Power interaction can be studied by investigating the terms of $\vec{E}_z \cdot \vec{J}^*$ which are obtained from Eq. (3-38) after taking the dot product of $\vec{E}_z$ and the conjugate of (3-38). We can see from Fig. (3-21) that the power term involving the space-charge of the beam has a negative real part indicating that this term gives energy to the system while the term involving the a.c. beam velocity has a positive real part indicating that it absorbs energy from the system. The terms involving the displacement current, the plasma current, and

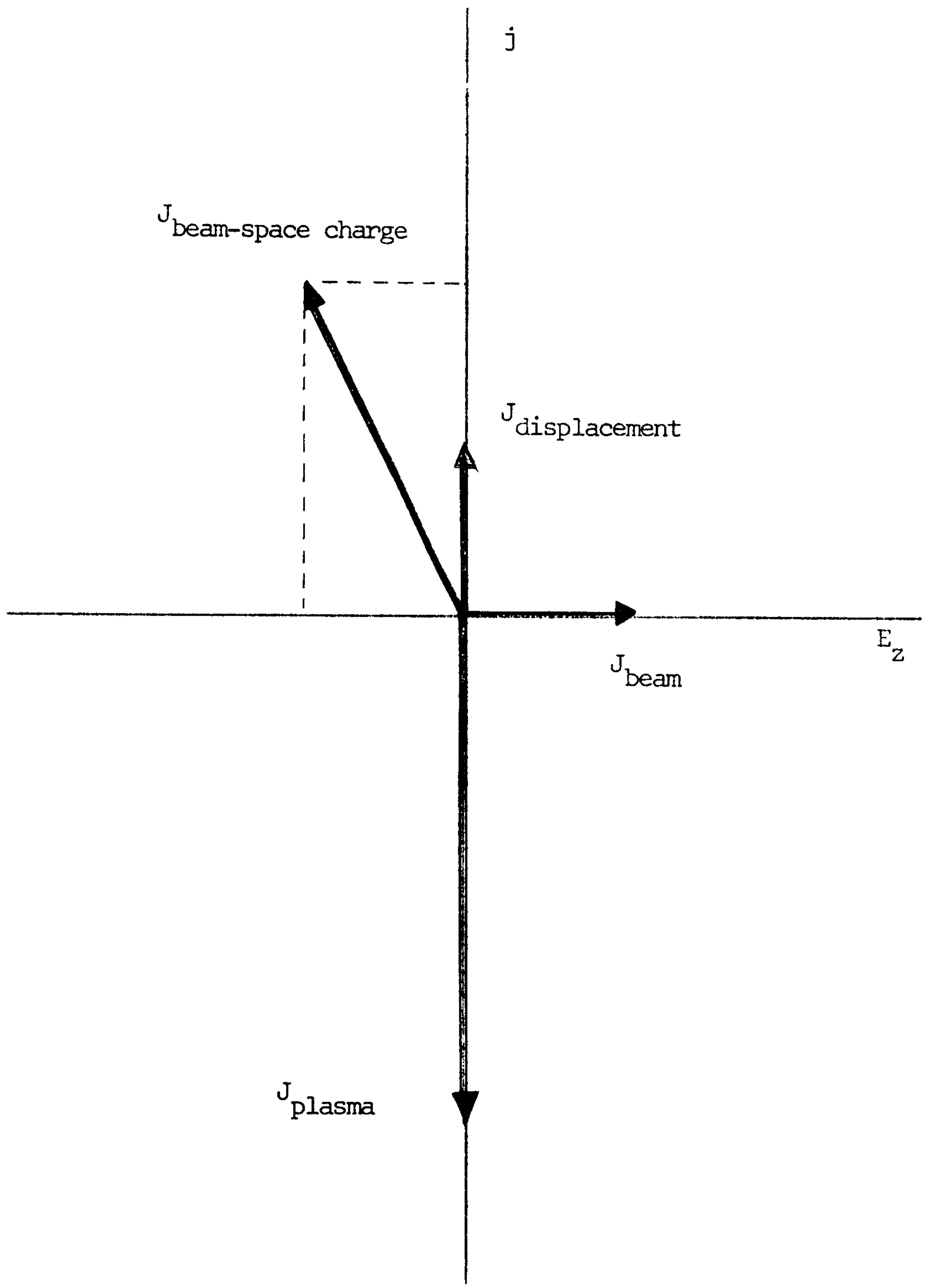

Fig. (3-21) Beam-Plasma Current Phasor Diagram the imaginary portion of the beam space-charge current are energy storage terms. It should be noted that the energy stored in the plasma and free space per unit time is the negative of that stored by the bunched electrons of the beam so that a resonance exists between these two systems. At the same time, the energy transferred from the bunched electron beam is equal to the energy picked up by the a.c. beam velocity.

At first, there seems to be no net gain of energy in the system, however, the increase of a.c. beam velocity is also related to the electric field $\vec{E}_z$ through the force equation, (3-12). Thus, an increase in electric field is also produced so that the total energy of the system increases. The beam space-charge, then, provides the gain by interacting with the electric field with a loss of its d.c. velocity, providing the necessary energy. We have not included anywhere in this discussion, the gradient in d.c. beam velocity due to this energy transfer. It has been assumed that the energy transfer per unit length is small so that there is little change in the d.c. beam velocity. The beam interaction process is the same as that which produces gain in conventional traveling-wave tubes except that the plasma, rather than a metallic helix, provides the "slow-wave structure."

A major distinction between the plasma and helix "slow-wave structures" is that the plasma may enhance the gain mechanism. The plasma is necessary for this gain system because it provides the inductive energy storage system that allows the electric field to become quite large. In fact, the reason that this system predicts infinite gain near the plasma frequency is that near reasonance, $\omega = \omega_p$, the electric field associated with the plasma oscillations become very large and, therefore, the beam interaction with the field is greatly enhanced. Introducing collision frequencies in the plasma, temperatures in the plasma, or finite dimensions of the system will not allow the plasma oscillations to approach a very large level near the plasma frequency and thus the electric field and the gain per unit length will remain finite.

3.4 Transverse Waves-- $\vec{k} \cdot \vec{E} = 0$

If $\vec{k} \cdot \vec{E} = 0$, there is no volume charge for the electric lines of force to terminate on so that the electric field is solenoidal and the pressures are zero. The equation governing the electric field, (3-23), reduces to:

$$\left[(k_o^2 - k^2)\, \bar{\bar{I}} - \frac{k_o^2\, \omega_p^2}{\omega} \left\{ \bar{\bar{C}} - \vec{k}\,\vec{k}\, \frac{u_p^2}{\omega} \right\}^{-1} - k_o^2\, \frac{\omega_b^2}{\omega^2} (\omega - \vec{V}_b \cdot \vec{k}) \left\{ \bar{\bar{I}} + \frac{\vec{V}_b\, \vec{k}}{(\omega - \vec{V}_b \cdot \vec{k})} \right\} \cdot \left\{ D - \frac{\vec{k}\,\vec{k}\, u_b^2}{(\omega - \vec{V}_b \cdot \vec{k})} \right\}^{-1} \cdot \left\{ \bar{\bar{I}} + \frac{\vec{k}\, \vec{V}_b}{(\omega - \vec{V}_b \cdot \vec{k})} \right\} \right] \cdot \vec{E} = 0 \quad . \tag{3-39}$$

We shall again choose a special case in which $\vec{k}$ is directed along the positive z axis. The dispersion relation is given by the condition that the cofactors of the third column elements of the matrix multiplying $\vec{E}$ be zero. Using the relations obtained from the longitudinal waves section, Eqs. (3-24), (3-27), (3-29), and (3-31); we obtain the single relation

$$\left[(k_o^2 - k^2) - k_o^2 \frac{\omega_p^2}{\omega}\left(\frac{\Omega_p}{\Omega_p^2 - \Omega^2}\right) - k_o^2\, \omega_b^2 \frac{(\omega - V_b k_z)}{\omega^2}\left(\frac{\Omega_b}{\Omega_b^2 - \Omega^2}\right)\right]^2$$

$$- \left[k_o^2 \frac{\omega_p^2}{\omega}\left(\frac{\Omega}{\Omega_p^2 - \Omega^2}\right) - k_o^2 \frac{\omega_b^2}{\omega^2}(\omega - V_b\, k_z)\left(\frac{\Omega}{\Omega_b^2 - \Omega^2}\right)\right]^2 = 0 \qquad (3\text{-}40)$$

or

$$k_z^2 = k_o^2 \left\{1 - \frac{\omega_p^2}{\omega(\Omega_p \pm \Omega)} - \frac{\omega_b^2(\omega - V_b\, k_z)}{\omega^2(\Omega_b \pm \Omega)}\right\} . \qquad (3\text{-}41)$$

Equation (3-41) describes the left and right hand circularly polarized waves that propagate in the beam-plasma system where the (+) sign corresponds to left-hand polarization or counterclockwise rotation about the d.c. magnetic field and the (-) sign corresponds to right-hand polarization or a clockwise rotation of the electric vector. If the term $\vec{V}_b \times \mu_o \vec{H}$ had been neglected as in a quasi-static approach, the last term in Eq. (3-41) would have been

$$- \frac{k_o^2\, \omega_b^2}{\omega(\Omega_b \pm \Omega)} . \qquad (3\text{-}42)$$

In any event, there is no gain mechanism in this transverse field case, so that no growing waves can be excited. This fact is a result of there being no a.c. space-charge current which may interact with the electric field.

Chapter 4

BOUNDED BEAM-PLASMA INTERACTION

4.1 Introduction

We have shown in Chapter 2 that the field quantities of the beam-plasma system can be derived from a set of scalar potential functions, E_z , H_z , P_p , and P_b ; which are the z directed electric and magnetic fields and the plasma and beam pressures respectively. In general for small signal conditions, we have seen that the equations for these potentials are a set of second order, linear, coupled equations which may have coupled boundary conditions.

The method of solution that will be employed here consists of decoupling the equations by means of a similarity transformation, which, in general, causes the boundary conditions to be coupled. We essentially change the basis functions by which we describe the physical system to a new set of basis functions which are uncoupled in the set of second order equations. This approach to the problem is similar to that used by Bresler and Marcuvitz,[22] Sancer,[23] and more recently by Chen and Cheng.[24]

The method is nearly impossible to carry out algebraically for a general system and computers are necessary to determine the eigenvalues of the basis potential functions and the propagation constants. Once the eigenvalues and propagation constants are found, however, the exact solution of the problem is exhibited in terms of appropriate eigenfunctions.

Other techniques, such as a perturbation method which treats the coupling terms as forcing functions for the equation, could be used to solve the coupled set of equations; but this becomes somewhat tedious when there is coupling between many of the basis functions. Such a technique has been used by Nield[25] in investigating guided waves in a warm plasma filled waveguide. We shall now describe the transformation approach to solve the general bounded, beam-plasma interaction problem where the beam and plasma may occupy different regions of space.

4.2 The Similarity Transformation

The four homogeneous differential equations derived in Appendix B, Eqs. (2-73)-(2-76), can be expressed in terms of a four-vector and the coupling exhibited by a coupling matrix. The expression is concisely:

$$\nabla_t^2\{X\} + \bar{\bar{F}} \cdot \{X\} = 0$$

or

$$\nabla_t^2 \begin{Bmatrix} E_z \\ H_z \\ P_p \\ P_b \end{Bmatrix} + \begin{Bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{41} & \cdots & \cdots & f_{44} \end{Bmatrix} \begin{Bmatrix} E_z \\ H_z \\ P_p \\ P_b \end{Bmatrix} = 0 \qquad (4\text{-}1)$$

subject to the boundary conditions. If we can find a transformation matrix, $\bar{\bar{S}}$, such that $\bar{\bar{S}}^{-1} \cdot \bar{\bar{F}} \cdot \bar{\bar{S}} = \bar{\bar{\Lambda}}$, where $\bar{\bar{F}}$ is the coupling matrix of Eq. (4-1) and $\overline{\overline{\Lambda}}$ is the diagonal eigenvalue matrix of $\overline{\overline{F}}$; then we can transform the four-vector $\vec{X}$ into a new vector $\vec{U}$ by the relation $\vec{X} = \overline{\overline{S}} \cdot \vec{U}$ with the result that the transformed set of equations are uncoupled. Following this procedure for Eq. (4-1) gives:

$$\begin{aligned} &\nabla_t^2\{X\} + \overline{\overline{F}} \cdot \{X\} = 0 \\ &\vec{X} = \overline{\overline{S}} \cdot \vec{U} \\ &\nabla_t^2\{\overline{\overline{S}} \cdot \vec{U}\} + \overline{\overline{F}} \cdot \{\overline{\overline{S}} \cdot \vec{U}\} = 0 \end{aligned} \tag{4-2}$$

Operating on the left by $\overline{\overline{S}}^{-1}$ and realizing that $\overline{\overline{S}}^{-1}$ commutes with the transverse Laplacian operator as long as $\overline{\overline{S}}$ is at most a function of z coordinate, we obtain:

$$\nabla_t^2\{\overline{\overline{S}}^{-1} \cdot \overline{\overline{S}} \cdot \vec{U}\} + \{\overline{\overline{S}}^{-1} \cdot \overline{\overline{F}} \cdot \overline{\overline{S}}\} \cdot \{U\} = 0 \tag{4-3}$$

or

$$\nabla_t^2\{U\} + \overline{\overline{\Lambda}} \cdot \{U\} = 0 \tag{4-4}$$

subject to the boundary conditions that are determined by the boundary conditions on $\vec{X}$ and the transformation $\vec{X} = \overline{\overline{S}} \cdot \vec{U}$. In its explicit form, Eq. (4-4) is written:

$$\nabla_t^2 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \lambda_3^2 & \\ 0 & & & \lambda_4^2 \end{Bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = 0 \tag{4-5}$$

where the λ^2's are the eigenvalues of the matrix $\bar{\bar{F}}$. In this uncoupled form, we can choose the set of eigenfunctions U_i and use the boundary conditions to select the allowable eigenfunctions and eigenvalues for the problem.

This diagonalization procedure can always be carried out as long as the matrix $\bar{\bar{F}}$ is <u>semi-simple</u>--that is, as long as the matrix $\bar{\bar{F}}$ has a complete set of eigenvectors regardless of degeneracy of eigenvalues. If $\bar{\bar{F}}$ is not <u>semi-simple</u> then the transformed matrix $\bar{\bar{\Lambda}}$ can be put in the Jordan canonical form with the eigenvalues of $\bar{\bar{F}}$ on the diagonal and ones or zero's in the position adjacent to two equal eigenvalues. Thus, in the Jordan canonical form, $\bar{\bar{\Lambda}}$ might appear as

$$\bar{\bar{\Lambda}} = \begin{bmatrix} \lambda_1^2 & 1 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_3^2 & 0 \\ 0 & 0 & 0 & \lambda_4^2 \end{bmatrix} \qquad (4\text{-}6)$$

The transformation matrix $\bar{\bar{S}}$ that must be used is the matrix formed by filling the columns of $\bar{\bar{S}}$ with vectors proportional to the eigenvectors of the matrix $\bar{\bar{F}}$. In the event that $\bar{\bar{F}}$ is not <u>semi-simple</u>, we can construct $\bar{\bar{S}}$ such that the set of vectors is linearly independent with the result that the transformed matrix, $\bar{\bar{\Lambda}}$, is in Jordan canonical form. A more thorough discussion of this problem is given by Pease.[26]

If the eigenvalues of $\bar{\bar{F}}$ are not distinct and the matrix $\bar{\bar{F}}$ is not semi-simple, then a perturbation method may be used to solve a set of equations as in (4-6) where the coupling term is treated as a perturbing source function.

4.3 Beam-Plasma Region

For the region of space occupied by both the beam and plasma, the set of relations describing the interaction is written in the following form:

$$\nabla_t^2 \begin{Bmatrix} E_z \\ H_z \\ P_p \\ P_b \end{Bmatrix} + \begin{Bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ f_{14} & \cdots & \cdots & f_{44} \end{Bmatrix} \begin{Bmatrix} E_z \\ H_z \\ P_p \\ P_b \end{Bmatrix} = 0 \qquad (4\text{-}7)$$

where

$$f_{11} = k_o^2 \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)} \right) - k^2$$

$$f_{12} = 0$$

$$f_{13} = j\,\frac{k\,e}{m\,\varepsilon_o} \left(\frac{\omega\,\mu_o\,\varepsilon_o}{\omega - j\nu_p} - \frac{1}{u_p^2} \right)$$

$$f_{14} = j\,\frac{k\,e}{m\varepsilon_o} \left(\frac{\omega\,\mu_o\,\varepsilon_o}{\omega - kV_b - j\nu_b} - \frac{1}{u_b^2} \right) + j\,\frac{\omega\,\mu_o\,e\,V_b}{m\,u_b^2}$$

$$f_{21} = -\,jk\,\varepsilon_o\Omega \left\{ \frac{\omega_p^2}{(\omega - j\nu_p)^2} + \frac{\omega_b^2}{(\omega - kV_b - j\nu_b)^2} \right\}$$

$$f_{22} = k_o^2 \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2(\omega - kV_b)}{\omega^2(\omega - kV_b - j\nu_b)}\right) - k^2$$

$$f_{23} = \frac{e\Omega \left(\omega - \frac{k^2\, u_p^2}{(\omega - j\nu_p)}\right)}{(\omega - j\nu_p)\; m\; u_p^2}$$

$$f_{24} = \frac{e\Omega \left(\omega - kV_b - \frac{k^2\, u_b^2}{(\omega - kV_b - j\nu_b)}\right)}{(\omega - kV_b - j\nu_b)\; m\; u_b^2}$$

$$f_{31} = j\, \frac{k\; \varepsilon_o\; \Omega^2\; \omega_p^2\, \frac{m}{e}}{(\omega - j\nu_p)^2}$$

$$f_{32} = \frac{m}{e} \left(\frac{\omega\; \mu_o\; \varepsilon_o\; \omega_p^2}{(\omega - j\nu_p)}\right)$$

$$f_{33} = \frac{1}{u_p^2} \left\{\omega(\omega - j\nu_p) - k^2 u_p^2 - \omega_p^2 - \frac{\Omega^2}{(\omega - j\nu_p)^2}\, [\omega(\omega - j\nu_p) - k^2 u_p^2]\right\}$$

$$f_{34} = -\, \frac{\omega_p^2}{u_b^2}$$

$$f_{41} = j\, \left\{\frac{k\Omega^2\; m\; \omega_b^2\; \varepsilon_o}{e(\omega - kV_b - j\nu_b)^2} - \omega_b^2\, \frac{m}{e}\, V_b \mu_o \varepsilon_o^2 \omega \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)}\right)\right\}$$

$$f_{42} = \frac{(\omega - kV_b)\; \omega_b^2\; \mu_o\; \varepsilon_o\; m\; \Omega}{e(\omega - kV_b - j\nu_b)}$$

$$f_{43} = \frac{\omega_b^2 \, kV_b \, \mu_o \, \varepsilon_o}{(\omega - j\nu_p)} - \frac{\omega_b^2}{u_p^2}$$

$$f_{44} = \frac{1}{u_b^2} \left\{ (\omega-kV_b)(\omega-kV_b-j\nu_b) - k^2 \, u_b^2 - \omega_b^2 \right.$$

$$- \Omega^2 \, \frac{[(\omega-kV_b)(\omega-kV_b-j\nu_b) - k^2 \, u_b^2]}{(\omega - kV_b - j\nu_b)^2}$$

$$\left. + \omega_b^2 \, V_b \, \mu_o \, \varepsilon_o \left(\frac{k \, u_b^2}{(\omega-kV_b-j\nu_b)} + V_b \right) \right\}$$

The eigenvalues of $\bar{\bar{F}}$ are found from the relation

$$\text{Determinant}(\bar{\bar{F}} - \lambda^2 \, \bar{\bar{I}}) = 0 \tag{4-8}$$

which gives a fourth order equation in λ^2 and can be solved by computer techniques to yield the four eigenvalues $\lambda_1^2 \, \ldots \, \lambda_4^2$.

Since we have to construct the columns of $\bar{\bar{S}}$ proportional to the eigenvectors of $\bar{\bar{F}}$, we can simply construct the vectors $\vec{S}_{ik}$ proportional to the cofactors of one row of $(\bar{\bar{F}} - \lambda_k^2 \, \bar{\bar{I}})$. This can be shown by an actual solution of the simultaneous equations for S_{ik} . Thus, one possible similarity transformation matrix might appear as

$$\bar{\bar{S}} = \begin{bmatrix} F_{11}(\lambda_1^2) & F_{11}(\lambda_2^2) & F_{11}(\lambda_3^2) & F_{11}(\lambda_4^2) \\ F_{12}(\lambda_1^2) & F_{12}(\lambda_2^2) & F_{12}(\lambda_3^2) & F_{12}(\lambda_4^2) \\ F_{13}(\lambda_1^2) & F_{13}(\lambda_2^2) & F_{13}(\lambda_3^2) & F_{13}(\lambda_4^2) \\ F_{14}(\lambda_1^2) & F_{14}(\lambda_2^2) & F_{14}(\lambda_3^2) & F_{14}(\lambda_4^2) \end{bmatrix} \tag{4-9}$$

where the $F_{ij}(\lambda_k^2)$ are the i,j th cofactors of $(\bar{\bar{F}} - \lambda_k^2 \bar{\bar{I}})$. It is clear that we cannot construct the columns of $\bar{\bar{S}}$ in the above manner if $\lambda_\alpha^2 = \lambda_\beta^2$ or one of the eigenvalues is zero since $\bar{\bar{S}}$ would then have no inverse. Thus, if $\bar{\bar{S}}$ has a repeated eigenvalue, λ_α^2 , we must construct the first vector in the manner described above and construct the remaining vectors for λ_α^2 such that the $\bar{\bar{S}}$ matrix is composed of a set of linearly independent vectors. Under the transformation given by Eqs. (4-2)-(4-4), then, we obtain the transformed set of equations

$$\nabla_t^2 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} + \begin{Bmatrix} \lambda_1^2 & & 0 & \\ & \lambda_2^2 & & \\ & & \lambda_3^2 & \\ & 0 & & \lambda_4^2 \end{Bmatrix} \cdot \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = 0 \tag{4-10}$$

subject to the boundary conditions given by $\vec{X} = \bar{\bar{S}} \cdot \vec{U}$ and the boundary conditions on $\vec{X}$. We may pick, then, the class of functions that generally satisfy Eq. (4-10) and use the boundary conditions to restrict these functions and the eigenvalues to an allowable set.

4.4 Plasma Region

The set of second order differential equations for the plasma region is obtained from the general beam-plasma set by placing all plasma quantities equal to zero. The set of equations becomes:

$$\nabla_t^2 \begin{Bmatrix} E_z \\ H_z \\ P_p \end{Bmatrix} + \begin{Bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{Bmatrix} \cdot \begin{Bmatrix} E_z \\ H_z \\ P_p \end{Bmatrix} = 0 \qquad (4\text{-}11)$$

or

$$\nabla_t^2\{Y\} + \bar{\bar{G}} \cdot \{Y\} = 0 \quad ,$$

where

$$g_{11} = k_o^2\left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)}\right) - k^2$$

$$g_{12} = 0$$

$$g_{13} = j\,\frac{k\ e}{m\varepsilon_o}\left(\frac{\omega\ \mu_o\ \varepsilon_o}{\omega - j\nu_p} - \frac{1}{u_p^2}\right)$$

$$g_{21} = -\ jk\ \varepsilon_o\Omega\left(\frac{\omega_p^2}{(\omega - j\nu_p)^2}\right)$$

$$g_{22} = k_o^2\left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)}\right) - k^2$$

$$g_{23} = \frac{e\Omega\left(\omega - \dfrac{k^2\ u_p^2}{(\omega - j\nu_p)}\right)}{(\omega - j\nu_p)\ m\ u_p^2}$$

$$g_{31} = \frac{jk\ \varepsilon_o\ \Omega^2\ \omega_p^2\ m}{e(\omega - j\nu_p)^2}$$

$$g_{32} = \frac{m}{e}\,\Omega \left(\frac{\omega\ \mu_o\ \varepsilon_o\ \omega_p^2}{\omega - j\nu_p}\right)$$

$$g_{33} = \frac{1}{u_p^2} \left(\omega(\omega - j\nu_p) - k^2 u_p^2 - \omega_p^2\right.$$

$$\left. - \frac{\Omega^2}{(\omega - j\nu_p)^2} [\omega(\omega - j\nu_p) - k^2\ u_p^2]\right) \quad .$$

The similarity transformation $\bar{\bar{T}}$ is obtained from the cofactors of $\bar{\bar{G}}$. One possible transformation is

$$\bar{\bar{T}} = \begin{bmatrix} G_{11}(\mu_1^2) & G_{11}(\mu_2^2) & G_{13}(\mu_3^2) \\ G_{12}(\mu_1^2) & G_{12}(\mu_2^2) & G_{12}(\mu_3^2) \\ G_{13}(\mu_1^2) & G_{13}(\mu_2^2) & G_{13}(\mu_3^2) \end{bmatrix} \tag{4-12}$$

where the $G_{ij}(\mu_k^2)$ are the i,j th cofactors of $(\bar{\bar{G}} - \mu_k^2\ \bar{\bar{I}})$ and $\mu_1^2 \ldots \mu_3^2$ are the eigenvalues of $\bar{\bar{G}}$ obtained from

$$\text{Determinant}(\bar{\bar{G}} - \mu^2\ \bar{\bar{I}}) = 0 \quad . \tag{4-13}$$

The transformation of (4-11) is obtained by using the similarity transformation $\bar{\bar{T}}$ such that

$$\vec{Y} = \bar{\bar{T}} \cdot \vec{V} \tag{4-14}$$

where $\vec{Y}$ is the potential function three-vector

$$\vec{Y} = \begin{Bmatrix} E_z \\ H_z \\ P_p \end{Bmatrix} . \tag{4-15}$$

The resultant transformation gives:

$$\nabla_t^2 \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} + \begin{Bmatrix} \mu_1^2 & & 0 \\ & \mu_2^2 & \\ 0 & & \mu_3^2 \end{Bmatrix} \cdot \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = 0 \tag{4-16}$$

provided of course that $\overline{\overline{G}}$ is <u>semi-simple</u>. We may now pick the class of functions satisfying (4-16) and use the boundary conditions determined by $\vec{Y} = \overline{\overline{T}} \cdot \vec{V}$ and the boundary conditions on $\vec{Y}$ to choose the allowable eigenfunctions and eigenvalues.

Chapter 5

FINITE BEAM-INFINITE PLASMA SYSTEM

5.1 Introduction

We now consider a particular example to illustrate the methods developed in the preceding chapters. We shall investigate the interaction of a finite beam in an infinite plasma where the beam and plasma are both warm and are assumed to have electron collisions with neutrals and ions of the background. No d.c. magnetic field acts in the problem and the a.c. magnetic field component, H_z , is assumed to be zero so that we are looking for the transverse magnetic (TM) modes that propagate in the system. In addition, the assumption is made that all potentials are axisymmetric and there is no variation of the beam-plasma quantities with respect to the polar angle θ .

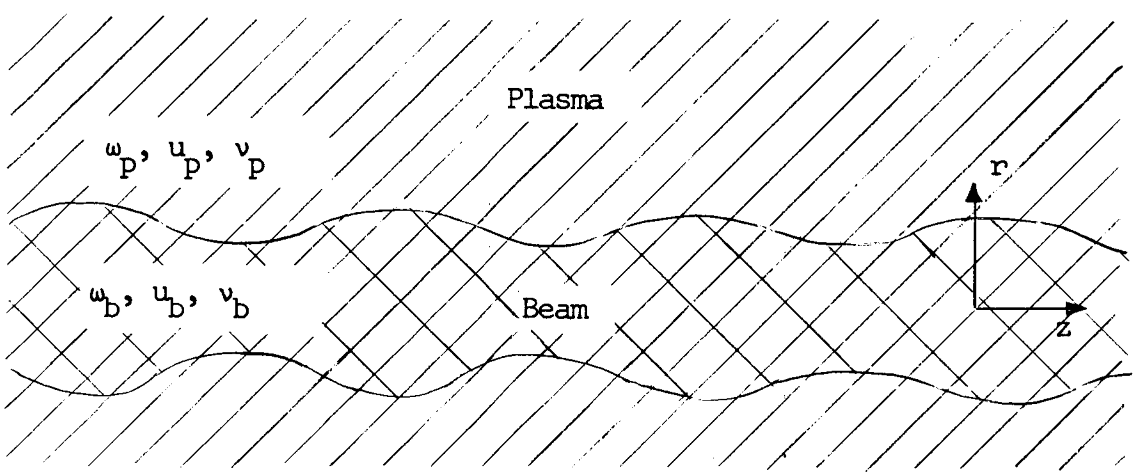

Fig. (5.1) BEAM-PLASMA SYSTEM

This problem is very similar to the one that was previously investigated by Crawford[19] except that no quasi-static approximation is made here and the beam is assumed to be warm. The electron beam is cylindrical in shape and interpenetrates the plasma background.

5.2 Beam-Plasma Region

For the beam-plasma region, we must diagonalize the coupling matrix as in Chapter 4 so that the equations describing the beam-plasma region become:

$$\nabla_t^2 \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} + \begin{Bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{Bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = 0 \tag{5-1}$$

and

$$\vec{X} \equiv \begin{Bmatrix} E_z \\ P_p \\ P_b \end{Bmatrix} = \bar{\bar{S}} \cdot \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \bar{\bar{S}} \cdot \vec{U} \tag{5-2}$$

where $\bar{\bar{S}}$ is the similarity transform defined in Chapter 4 and E_z, P_p, P_b are the potential functions for our problem. The form of (5-1) is obtained if the eigenvalues are distinct or the coupling matrix, $\bar{\bar{F}}$, is <u>semi-simple</u>.

In a cylindrical system where the potentials have no θ variation, the solutions to (5-1) are of the form:

$$U_i = a_i J_o(\lambda_i r) + b_i Y_o(\lambda_i r) \tag{5-3}$$

where a_i and b_i are amplitude constants, J_o and Y_o are the zeroth order Bessel functions of the first and second kind, respectively; and the λ_i are the square roots of the eigenvalues of $\overline{\overline{F}}$ which in general are complex.

5.3 Plasma Region

The equations in the plasma region are written in a form similar to those of the beam-plasma region.

$$\nabla_t^2 \begin{Bmatrix} V_1 \\ \\ V_2 \end{Bmatrix} + \begin{Bmatrix} \mu_1^2 & 0 \\ \\ 0 & \mu_2^2 \end{Bmatrix} \begin{Bmatrix} V_1 \\ \\ V_2 \end{Bmatrix} = 0 \tag{5-4}$$

and

$$\vec{Y} = \begin{Bmatrix} E_z \\ \\ P_p \end{Bmatrix} = \overline{\overline{T}} \cdot \begin{Bmatrix} V_1 \\ \\ V_2 \end{Bmatrix} = \overline{\overline{T}} \cdot \vec{V} \tag{5-5}$$

where $\overline{\overline{T}}$ is the similarity transform matrix as defined in Chapter 4. The solutions for (5-4) are chosen to be of the form:

$$V_i = c_i H_o^1(\mu_i r) + d_i H_o^2(\mu_i r) \tag{5-6}$$

where H_o^1 and H_o^2 are the zeroth order Hankel functions of the first and second kind, respectively, and the μ_i are the square roots of the eigenvalues of the coupling matrix $\bar{\bar{G}}$.

5.4 Boundary Conditions

The boundary conditions we shall apply to the system are boundedness of the potential functions at the origin and the radiation condition for large radii and the boundary conditions at the beam-plasma--plasma interface which are derived in Appendix C. These conditions are written in the following manner:

Boundedness at the Origin

$$\left.\begin{matrix} E_z \\ P_p \\ P_b \end{matrix}\right\} - \text{Bounded at } r = 0 \tag{5-7}$$

Radiation Boundary Condition

$$\lim_{r \to \infty} r \frac{\partial}{\partial r}\left\{\begin{matrix} E_z \\ \\ P_p \end{matrix}\right\} + jk_r\left\{\begin{matrix} E_z \\ \\ P_p \end{matrix}\right\} = 0 \tag{5-8}$$

where k_r is the radial propagation constant. The conditions at $r = a$ are:

$$\left.\begin{matrix} E_z \\ P_p \\ \vec{v}_p \end{matrix}\right\} - \text{Continuous at } r = a \tag{5-9}$$

$$\varepsilon_o \, E_r \Big|_{r=a-}^{r=a+} = \rho_{sb} \tag{5-10}$$

$$N_b \, ev_{br}(r=a-) = \frac{-\rho_{sb}}{j(\omega - kV_b)} \tag{5-11}$$

$$\vec{a}_r \times \vec{H} \Big|_{r=a-}^{r=a+} = \rho_{sb} \, \vec{V}_b \tag{5-12}$$

The condition that $\vec{v}_p$ be continuous at the boundary is actually the condition that v_{pr} be continuous since $v_{p\theta}$ is zero for this problem and the conditions that E_z and P_p be continuous forces v_{pz} to be continuous because of the force equation for the plasma electrons. The condition of boundedness at the origin restricts the solutions (5-3) to be zeroth order Bessel functions of the first kind only.

$$U_i = a_i \, J_o(\lambda_i \, r) \tag{5-13}$$

The radiation condition at large radius restricts the solutions (5-6) to be the form:

$$V_i = c_i \, H_o^1(\mu_i \, r) \tag{5-14}$$

or

$$V_i = d_i \, H_o^2(\mu_i \, r) \tag{5-15}$$

depending on the branch we choose for $(\mu_i^2)^{\frac{1}{2}}$, since $H_o^1(\mu_i \, r)$ decays while $H_o^2(\mu_i \, r)$ grows exponentially for the imaginary part of μ_i being positive and radius increasing. Thus, if the imaginary part of μ_i is positive we must choose the H_o^1 function and if it is negative, the H_o^2 function. We can choose to pick only the H_o^1 function for either branch, however, since

$$H_o^2(-\gamma) = -H_o^1(\gamma) \qquad (5\text{-}16)$$

so that if μ_i has a negative imaginary part, we shall choose the potential function in the following form:

$$V_i = -c_i H_o^1(-\mu_i r) \quad . \qquad (5\text{-}17)$$

The conditions at the boundary $r = a$ can be written in the following manner:

$$E_z(r=a+) = E_z(r=a-) \qquad (5\text{-}18)$$

$$P_p(r=a+) = P_p(r=a-) \qquad (5\text{-}19)$$

$$v_{pr}(r=a+) = v_{pr}(r=a-) \qquad (5\text{-}20)$$

$$\epsilon_o E_r(r=a+) - \epsilon_o E_r(r=a-) = \rho_{sb} \qquad (5\text{-}21)$$

$$N_b e\, v_{br}(r=a-) = -\frac{\rho_{sb}}{j(\omega - kV_b)} \qquad (5\text{-}22)$$

$$H_\theta(r=a+) - H_\theta(r=a-) = \rho_{sb} V_b \quad . \qquad (5\text{-}23)$$

Equations (5-21) and (5-22) give just one relation for v_{br} and E_r

so that we have five independent boundary conditions to be applied at the boundary $r = a$ and five unknown amplitude coefficients for the five potential functions U_i , V_i .

We can write the relation (5-2) for the potentials in the beam-plasma region in its expanded form.

$$
\begin{aligned}
E_z &= S_{11} U_1 + S_{12} U_2 + S_{13} U_3 = \sum_{j=1}^{3} S_{1j} U_j \\
P_p &= S_{21} U_1 + S_{22} U_2 + S_{23} U_3 = \sum_{j=1}^{3} S_{2j} U_j \\
P_b &= S_{31} U_1 + S_{32} U_2 + S_{33} U_3 = \sum_{j=1}^{3} S_{3j} U_j
\end{aligned}
\tag{5-24}
$$

Similarly for the plasma region, the relation (5-5) becomes:

$$
\begin{aligned}
E_z &= T_{14} V_4 + T_{15} V_5 = \sum_{j=4}^{5} T_{1j} V_j \\
P_p &= T_{24} V_4 + T_{25} V_5 = \sum_{j=4}^{5} T_{2j} V_j \quad .
\end{aligned}
\tag{5-25}
$$

The set of boundary conditions for the beam-plasma--plasma interface can be obtained in terms of the potential functions U_i , V_i by using the relations (5-18)-(5-23), (5-24), (5-25) and the relationships derived in Chapter 2 for E_r , H_θ , v_{pr} , and v_{br} . This set of equations is written in its final form so that we have a set of five homogeneous equations in terms of five unknown amplitude coefficients.

$$a_1 P_{11} J_o(\lambda_1 a) + a_2 P_{12} J_o(\lambda_2 a) + a_3 P_{13} J_o(\lambda_3 a) + C_4 P_{14} H_o^1(\mu_4 a) + C_5 P_{15} H_o^1(\mu_5 a) = 0$$

$$a_1 P_{21} J_o(\lambda_1 a) + \dots\dots\dots\dots \quad C_5 P_{25} H_o^1(\mu_5 a) = 0$$

$$a_1 P_{31} \lambda_1 J_1(\lambda_1 a) + a_2 P_{32} \lambda_2 J_1(\lambda_2 a) + \dots\dots\dots \quad C_5 P_{35} H_1^1(\mu_5 a) = 0$$

$$\vdots$$

$$a_1 P_{51} \lambda_1 J_1(\lambda_1 a) + a_2 P_{52} \lambda_2 J_1(\lambda_2 a) + \dots\dots\dots \quad C_5 P_{55} H_1^1(\mu_5 a) = 0 \ . \tag{5-26}$$

We have not as yet determined the eigenvalues which satisfy this set of equations for arbitrary amplitude coefficients. The solutions for the eigenvalues can be obtained by requiring the condition that the determinant of the matrix multiplying the coefficients, a_i, c_i , must be zero in order to have non-trivial solutions for these coefficients.

The procedure for determining the correct values of propagation constant and eigenvalues is then as follows:

(1) Pick a value of the propagation constant k , which has both real and imaginary parts

(2) Calculate the eigenvalues λ_i , μ_i

(3) Construct the boundary condition matrix whose elements multiply the coefficients a_i , c_i

(4) Find the determinant of the matrix which must be zero if there is to be a solution for the problem.

This procedure must be an iteration scheme since one must change the input k such that the determinant vanishes. A computer program has been written to carry out these iterations and the final results for the finite beam-infinite plasma system are given in Figs. (5-2)-(5-4). A detailed description of the computer program is given in Appendix E.

For curve (1) shown in Figs. (5-2) and (5-3), the "temperature" of plasma electrons is of the order of .03 volts while that of the beam is roughly .004 volts. The d.c. beam velocity corresponds to an accelerating potential of about .8 volts and $V_b/u_p = 5$. This ratio for the unbounded system corresponds very nearly to the maximum gain condition.

The parameters chosen, represent a very slow beam in a relatively cool plasma. A comparison of the solutions for these parameters and the corresponding solutions of Chapter 3 indicates that the gain curves for the bounded system is considerably less broadband and that the finite radius beam system has less maximum gain.

The solutions of the determinental equation indicated that there were a number of solutions which satisfied the boundary condition matrix for each set of parameters, however, it was found that only one solution corresponded to wave phase velocities $\left(\frac{\omega}{k_{real}}\right)$ which were slightly less than the beam velocity V_b and could reasonably represent a traveling-wave instability.

Computations were performed for an infinite plasma-finite beam system with parameters corresponding to those used by Crawford and Cannara[29] in a quasi-static cold beam analysis which also neglected collisions. It was found that two solutions with both positive real and imaginary propagation parts exist as was demonstrated by Crawford and Cannara. However, it was observed that both solutions had phase velocities slightly larger than the beam velocity. It is believed from a physical point of view that these waves could not represent traveling wave instabilities when the phase velocity is greater than the beam velocity. Self has pointed out that in the absence of collisions or temperatures of sufficient magnitude, the instabilities appear to be absolute or non-convective and can be stabilized by introducing sufficient collisions, such that only traveling wave instabilities exist.

In an attempt to see this effect, a small amount of collisions were introduced for both the beam and the plasma. It was found that the collisions did slow the phase velocity of <u>one</u> wave to a point where it always had a phase velocity less than or equal to the beam velocity. The second solution which had a smaller imaginary propagation constant k_i was not similarly affected by the collisions and its phase velocity was greater than the beam velocity for frequencies below the plasma frequency.

A comparison of the curve of Crawford and Cannara and the curve computed with the effects of beam temperature, a.c. magnetic field and collisions included is shown in Fig. (5-4).

It is to be noted here, that the collisions that were introduced were very small in magnitude and did not alter the gain curve so that the effects of collisions are seen only in slowing the wave slightly.

The curves shown for the quasi-static cold beam system and the exact analysis show good agreement for the real propagation constant, but the exact analysis indicates somewhat greater gain and broader bandwidth indicating that the effects of beam temperature or the a.c. magnetic field are important. It is expected that these quantities would have a sizable effect on the beam-plasma boundary conditions.

Thus, although the quasi-static zero beam temperature case is much simpler to analize and gives a very good indication of the shape of the propagation curve, it is felt that a more exact analysis is needed to explain magnitude behavior of the propagation constant.

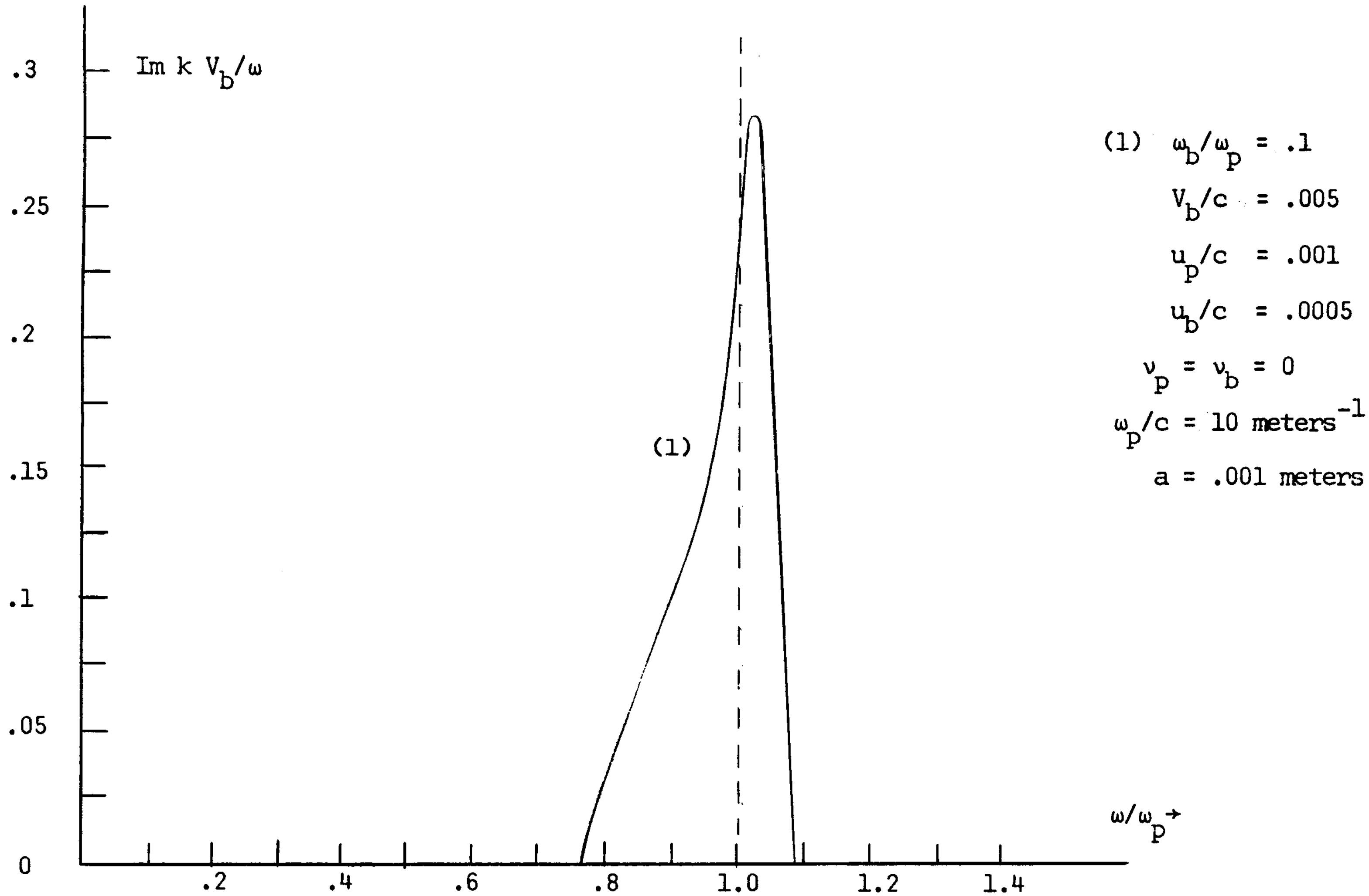

Fig. (5-2) IMAGINARY PART OF k V_b/ω VERSUS ω/ω_p BOUNDED BEAM

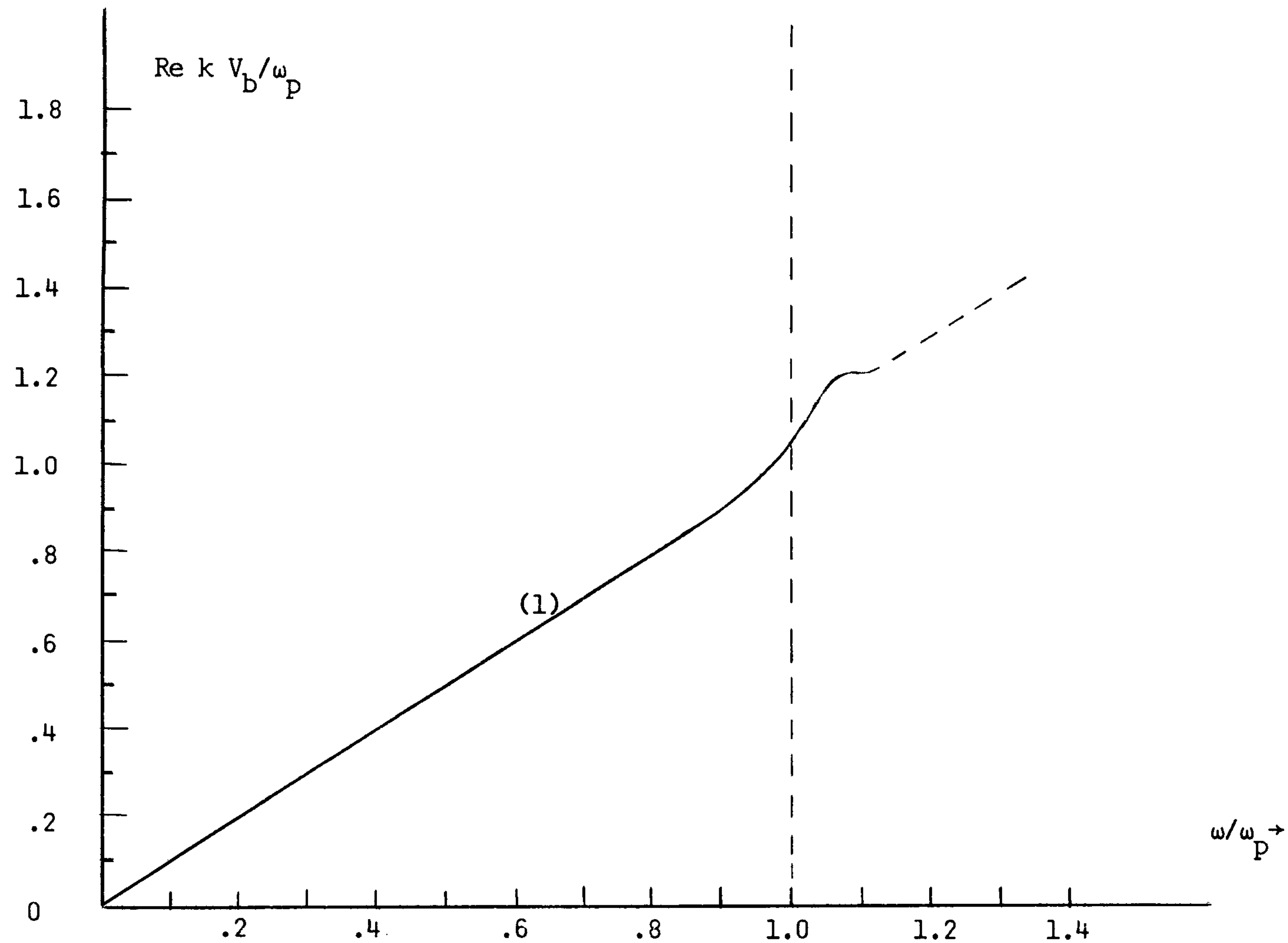

Fig. (5-3) REAL PART k V_b/ω_p VERSUS ω/ω_p

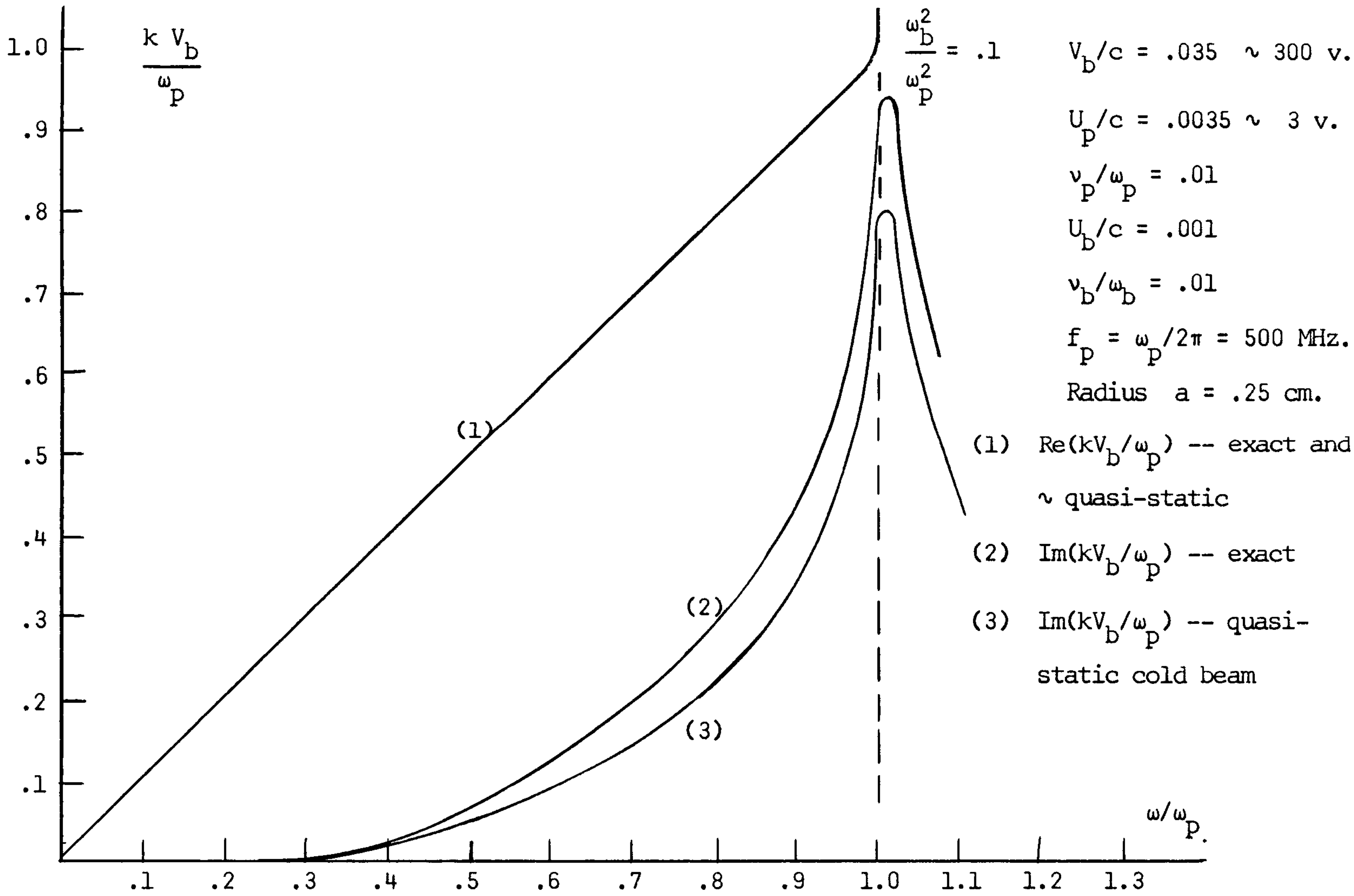

Fig. (5-4) COMPARISON OF EXACT AND QUASI-STATIC SOLUTIONS

Chapter 6

CONCLUSIONS

The theoretical development of Chapter 2 and the results indicated in Chapter 5 demonstrate that the coupled equations describing a beam-plasma system can be solved exactly with the aid of a high-speed digital computer and that the model chosen which includes the effects of collisions, temperatures, finite boundaries, a.c. magnetic fields and an external d.c. magnetic field is a self-consistent model. It should be noted, that this model is based on a small signal analysis and can not describe the beam-plasma interaction under conditions in which the a.c. perturbations are not small compared to the d.c. quantities. In addition, the d.c. magnetic field must be weak in order that the force term due to a pressure can be approximated by a gradient of the scalar pressure. One must keep in mind, also, that this model does not hold for all ranges of plasma and beam parameters or excitation frequencies since a number of assumptions were made concerning stationary ions and neutrals and the plasma and beam electrons were assumed to interact collectively through the a.c. electric and magnetic fields. One would expect that for extremely high plasma densities the model would not describe adequately the electron interaction.

The question arises as to whether a model chosen for beam plasma interaction would be drastically changed if the effects of finite gradients in the d.c. plasma densities were considered. Kollettis[30] has shown in a study of surface waves on a plasma column that a slow radial density variation of a plasma column can be accounted for satisfactorily by using an average electron density. One might expect the same conditions would hold in an electron beam-plasma experiment, since the potential solutions (excepting the beam pressure) are very nearly the same as a function of the transverse or radial direction since in both systems, the electromagnetic fields and the plasma pressures tend to maximize near the beam-plasma interface. A comparison of experimental results with theory would determine under which conditions the d.c. variations can be neglected. It seems very reasonable that in systems where the background plasma and the beam have rapid variation with radius, the effects of nonuniformity would be quite noticeable and the effects should be included if an accurate model is to be obtained.

The results presented graphically in Chapter 5 do not include effects of a d.c. magnetic field, however, the computer solution proceeds in exactly the same manner and is not any more difficult for the most general beam plasma problem. Searching for the roots that satisfy the boundary condition matrix is rather tedious and difficult, since the contours in the k plane generally have a number of sharp minima. Also, because of the way in which the simularity matrix is constructed, there are minima located at those points where two eigenvalues of the coupling matrix for the beam-plasma region are equal. These minima were taken to be erroneous solutions, although the possibility exists of having a solution which corresponds to repeated eigenvalues as described in Chapter 4. A search for all minima must be made and those solutions which are not reasonable solutions are discarded.

The reason underlying this investigation was to develop a general theory for the beam-plasma interaction which is tractible and includes most of the physical mechanisms of the beam plasma problem. A test of the validity of this model can only be obtained by a comparison of theory with experiment, but it is believed that including the effects of temperature and collisions in a non-quasi-static approach for both the beam and plasma quantities should more closely describe the physical problem.

APPENDIX A

A.C. POWER RELATIONS

The differential equations describing the beam-plasma wave interaction are rewritten for convenience.

$$\nabla \times \vec{E} + j\omega\, \mu_o\, \vec{H} = 0 \tag{A-1}$$

$$\nabla \times \vec{H} - j\omega\, \varepsilon_o\, \vec{E} + N_b\, e\, \vec{v}_b + N_p\, e\, \vec{v}_p + \frac{e\, P_b\, \vec{V}_b}{m u_b^2} = 0 \tag{A-2}$$

$$m\, N_p (j\omega + \nu_p)\, \vec{v}_p + N_p\, e(\vec{E} + \vec{v}_p \times \vec{B}_o) + \nabla P_p = 0 \tag{A-3}$$

$$m\, N_b (j\omega + \{V_b \cdot \nabla\} + \nu_b)\vec{v}_b + N_b\, e(\vec{E} + \vec{V}_b \times \mu_o \vec{H} + \vec{v}_b \times \vec{B}_o) + \nabla P_b = 0 \tag{A-4}$$

$$u_p^{\,2}\, m\, \nabla \cdot (N_p\, \vec{v}_p) + j\omega\, P_p = 0 \tag{A-5}$$

$$u_b^2\, m\, \nabla \cdot \left(N_b\, \vec{v}_b + \frac{P_b\, \vec{V}_b}{m u_b^2} \right) + j\omega\, P_b = 0 \tag{A-6}$$

The power relations are derived from the divergence of the Poynting Vector.

$$\begin{aligned} \nabla \cdot (\vec{E} \times \vec{H}^*) &= \vec{H}^* \cdot \nabla \times \vec{E} - \vec{E}^* \cdot \nabla \times \vec{H} \\ &= -\, j\omega\, \mu_o\, \vec{H} \cdot \vec{H}^* + j\omega\, \varepsilon_o\, \vec{E} \cdot \vec{E}^* - \vec{E} \cdot \vec{J}^* \end{aligned} \tag{A-7}$$

where

$$\vec{J} = -\left(N_b\ e\ \vec{v}_b + N_p\ e\ \vec{v}_p + \frac{e\ P_b\ \vec{V}_b}{mu_b^2} \right) \qquad \text{(A-8)}$$

and the asterisk, $*$, denotes the complex conjugate. The dot product of $\vec{E}$ and $\vec{J}^*$ is obtained by using Equation (A-8).

$$\vec{J}^* \cdot \vec{E} = -\left(N_b\ e\ \vec{v}_b^* \cdot \vec{E} + N_p\ e\ \vec{v}_p^* \cdot \vec{E} + \frac{e\ P_b^*\ \vec{V}_b \cdot \vec{E}}{m\ u_b^2} \right) \qquad \text{(A-9)}$$

The dot products of Equation (A-3) with $\vec{v}_p^*$ and Equation (A-4) with $\vec{v}_b^*$ and $\vec{V}_b$ give:

$$m\ N_p(j\omega + \nu_p)\ \vec{v}_p \cdot \vec{v}_p^* + N_p\ e\ \vec{v}_p^* \cdot \vec{E} + \vec{v}_p^* \cdot \nabla P_p = 0\ , \qquad \text{(A-10)}$$

$$m\ N_b(j\omega + \nu_b)\ \vec{v}_b \cdot \vec{v}_b^* + m\ N_b\ \vec{v}_b^* \cdot (\vec{V}_b \cdot \nabla)\ \vec{v}_b \qquad \text{(A-11)}$$

$$+ N_b\ e\ \vec{v}_b^* \cdot \vec{E} + N_b\ e\ \vec{v}_b^* \cdot (\vec{V}_b \times \mu_o\ \vec{H}) + \vec{v}_b^* \cdot \nabla P_b = 0\ ,$$

$$m\ N_b(j\omega + \nu_b)\ \vec{V}_b \cdot \vec{v}_b + m\ N_b\ \vec{V}_b \cdot (\vec{V}_b \cdot \nabla)\ \vec{v}_b \qquad \text{(A-12)}$$

$$+ N_b\ e\ \vec{V}_b \cdot \vec{E} + \vec{V}_b \cdot \nabla P_b = 0\ .$$

Equations (A-9)-(A-12) are now used to write the expression for $\vec{J}^* \cdot \vec{E}$:

$$\vec{J}^* \cdot \vec{E} = m\,N_p(j\omega + \nu_p)\,\vec{v}_p \cdot \vec{v}_p^* + \vec{v}_p^* \cdot \nabla P_p \tag{A-13}$$

$$+ m\,N_b(j\omega + \nu_b)\,\vec{v}_b \cdot \vec{v}_b^* + m\,N_b\,\vec{v}_b^* \cdot (\vec{V}_b \cdot \nabla)\,\vec{v}_b$$

$$+ N_b\,e\,\vec{v}_b^* \cdot (\vec{V}_b \times \mu_o\,\vec{H}) + \vec{v}_b^* \cdot \nabla P_b$$

$$+ \frac{P_b^*}{m\,u_b^2}\left\{m(j\omega + \nu_b)\vec{V}_b \cdot \vec{v}_b + m\,\vec{V}_b \cdot (\vec{V}_b \cdot \nabla)\vec{v}_b + \frac{\vec{V}_b \cdot \nabla P_b}{N_b}\right\} .$$

We can use the following relations to express (A-13) in a different form:

$$\vec{v}_p^* \cdot \nabla P_p = \nabla \cdot (P_p\,\vec{v}_p^*) - \frac{j\omega\,P_p\,P_p^*}{m\,u_p^2\,N_p} , \tag{A-14}$$

$$\vec{v}_b^* \cdot \nabla P_b + \frac{P_b^*\,\vec{V}_b \cdot \nabla P_b}{m\,u_b^2\,N_b} = \nabla \cdot \left(P_b\,\vec{v}_b^* + \frac{P_b\,P_b^*\,\vec{V}_b}{m\,u_b^2\,N_b}\right)$$

$$- P_b\,\nabla \cdot \left(\vec{v}_b^* + \frac{P_b^*\,\vec{V}_b}{m\,u_b^2\,N_b}\right) = \nabla \cdot \left(P_b\,\vec{v}_b^* + \frac{P_b\,P_b^*}{m\,u_b^2\,N_b}\right) - \frac{j\omega\,P_b\,P_b^*}{m\,u_b^2\,N_b} , \tag{A-15}$$

$$m\left\{N_b\,\vec{v}_b^* + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right\} \cdot \{\vec{V}_b \cdot \nabla\}\,\vec{v}_b + \frac{P_b^*}{m\,u_b^2}\{m(j\omega + \nu_b)\,\vec{V}_b \cdot \vec{v}_b\}$$

$$= \nabla \cdot \left\{m\left(N_b\,\vec{v}_b^* + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right)(\vec{V}_b \cdot \vec{v}_b)\right\} + \frac{P_b^*}{m\,u_b^2}\{m\,\nu_b\,\vec{V}_b \cdot \vec{v}_b\}$$

$$- m\left\{N_b\,\vec{v}_b^* + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right\} \cdot \{\vec{V}_b \times (\nabla \times \vec{v}_b)\} \;; \tag{A-16}$$

where Equation (A-6) was used to reduce (A-15) to its final form and the vector identity

$$\nabla(\vec{V}_b \cdot \vec{v}_b) = (\vec{V}_b \cdot \nabla)\,\vec{v}_b + \vec{V}_b \times (\nabla \times \vec{v}_b) \tag{A-17}$$

was used to obtain (A-16).

Equation (A-7) is rewritten with the aid of Equations (A-13)-(A-16).

$$\nabla \cdot \left\{ \vec{E} \times \vec{H}^* + P_p\,\vec{v}_p^{\,*} + P_b\left(\vec{v}_b^{\,*} + \frac{P_b^*\,\vec{V}_b}{m\,u_b^2\,N_b}\right) + m\left(N_b\,\vec{v}_b^{\,*} + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right)(\vec{V}_b \cdot \vec{v}_b)\right\}$$

$$= -\,j\omega\,\mu_o\,\vec{H} \cdot \vec{H}^* + j\omega\,\varepsilon_o\,\vec{E} \cdot \vec{E}^* - m\,N_p(j\omega + \nu_p)\,\vec{v}_p \cdot \vec{v}_p^{\,*}$$

$$-\,m\,N_b(j\omega + \nu_b)\vec{v}_b \cdot \vec{v}_b^{\,*} + \frac{j\omega\,P_p\,P_p^*}{m\,u_p^2\,N_p} + \frac{j\omega\,P_b\,P_b^*}{m\,u_b^2\,N_b} - \frac{P_b^*}{m\,u_b^2}(m\,\nu_b\,\vec{V}_b \cdot \vec{v}_b)$$

$$-\,N_b\,e\,\vec{v}_b^{\,*} \cdot (\vec{V}_b \times \mu_o\vec{H}) + m\left\{N_b\vec{v}_b^{\,*} + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right\} \cdot \{\vec{V}_b \times (\nabla \times \vec{v}_b)\} \tag{A-18}$$

The last term in Equation (A-18) can be expressed in a different manner with the aid of (A-4). The cross product of the veam velocity, $\vec{V}_b$, with the curl of Equation (A-4) gives:

$$\begin{aligned} &m\,N_b\{j\omega + (\vec{V}_b \cdot \nabla) + \nu_b\}\{\vec{V}_b \times \nabla \times \vec{v}_b\} - j\omega\,N_b\,e(\vec{V}_b \times \mu_o\,\vec{H}) \\ &\quad -\,N_b\,e(\vec{V}_b \cdot \nabla)(\vec{V}_b \times \mu_o\,\vec{H}) + N_b\,e(\vec{B}_o \cdot \nabla)(\vec{V}_b \times \vec{v}_b) = 0 \;. \end{aligned} \tag{A-19}$$

The dot product of (A-19) with $\vec{v}_b^{\,*}$ is written:

$$m\,N_b(j\omega + \nu_b)\,\vec{v}_b^{\,*} \cdot (\vec{V}_b \times \nabla \times \vec{v}_b) + m\,N_b\,\vec{v}_b^{\,*} \cdot (\vec{V}_b \cdot \nabla)(\vec{V}_b \times \nabla \times \vec{v}_b)$$

$$- j\omega\,N_b\,e\,\vec{v}_b^{\,*} \cdot (\vec{V}_b \times \mu_o \vec{H}) - N_b\,e\,\vec{v}_b^{\,*} \cdot (\vec{V}_b \cdot \nabla)(\vec{V}_b \times \mu_o \vec{H}) = 0\,. \tag{A-20}$$

If the beam collision frequency, ν_b, is zero; then

$$m\,N_b\,\vec{V}_b \times (\nabla \times \vec{v}_b) = N_b\,e\,(\vec{V}_b \times \mu_o\,\vec{H}) \tag{A-21}$$

since Equation (A-20) requires the two terms to differ only by a constant (or the operator $j\omega + \vec{V}_b \cdot \nabla = 0$ --a trivial result) if ν_b is zero. For the case in which beam collisions are neglected, then, the power relation reduces to:

$$\nabla \cdot \left\{ \vec{E} \times \vec{H}^* + P_p\,\vec{v}_p^{\,*} + P_b\left(\vec{v}_b^{\,*} + \frac{P_b^*\,\vec{V}_b}{m\,u_b^2\,N_b}\right) + m\left(N_b\,\vec{v}_b^{\,*} + \frac{P_b^*\,\vec{V}_b}{mu_b^2}\right)(\vec{V}_b \cdot \vec{v}_b)\right\}$$

$$= - j\omega\,\mu_o\,\vec{H} \cdot \vec{H}^* + j\omega\,\varepsilon_o\,\vec{E} \cdot \vec{E}^* - m\,N_p(j\omega + \nu_p)\,\vec{v}_p \cdot \vec{v}_p^{\,*} \tag{A-22}$$

$$- m\,N_b(j\omega)\,\vec{v}_b \cdot \vec{v}_b^{\,*} + \frac{j\omega\,P_p\,P_p^*}{m\,u_p^2\,N_p} + \frac{j\omega\,P_b\,P_b^*}{m\,u_b^2\,N_b}\,.$$

The general power relation, (A-18), can be rewritten using the relation:

$$\vec{J}_b \equiv - e\left(N_b\,\vec{v}_b + \frac{P_b\,\vec{V}_b}{mu_b^2}\right)\,. \tag{A-23}$$

The result is:

$$\nabla \cdot \left\{ \vec{E} \times \vec{H}^* + P_p \vec{v}_p^* - \frac{P_b \vec{J}_b^*}{N_b e} - \frac{m}{e} \vec{J}_b^* (\vec{V}_b \cdot \vec{v}_b) \right\}$$

$$= - j\omega \mu_o \vec{H} \cdot \vec{H}^* + j\omega \varepsilon_o \vec{E} \cdot E^* - m N_p (j\omega + \nu_p) \vec{v}_p \cdot \vec{v}_p^*$$

$$- m N_b (j\omega + \nu_b) \vec{v}_b \cdot \vec{v}_b^* + \frac{j\omega P_p P_p^*}{m u_p^2 N_p} + \frac{j\omega P_b P_b^*}{m u_b^2 N_b} \tag{A-24}$$

$$- P_b^* \frac{(m \nu_b \vec{V}_b \cdot \vec{v}_b)}{m u_b^2} + \vec{J}_b^* \cdot (\vec{V}_b \times \mu_o \vec{H}) + \frac{m}{e} (\vec{V}_b \times \vec{J}_b^*) \cdot (\nabla \times \vec{v}_b) .$$

Equation (A-24) is of the form one usually sees in the description of traveling wave tubes.[27]

If we are dealing with longitudinal wave solutions, then $\nabla \times \vec{E} = - j\omega \mu_o \vec{H} = 0$, and the power relation is obtained from (A-24) by setting $\vec{H}$ equal to zero. The relation is actually derived from Equation (A-2) but the result is the same one obtained by setting $\vec{H} = 0$ in (A-24).

The terms on the right hand side of (A-24) are interpreted in the following manner:

(1) The first six terms correspond to time rate of change of magnetic, electric, plasma electron kinetic, beam electron kinetic, plasma accoustic potential, and beam accoustic potential energies. Losses of kinetic energy correspond to terms involving ν_p and ν_b .

(2) The term $\frac{-P_b^*}{m\,u_b^2}(m\,\nu_b\,\vec{V}_b \cdot \vec{v}_b)$ is a power transfer term which may produce a gain depending on the phases of the beam pressure and beam velocity.

(3) $\vec{J}_b^* \cdot (\vec{V}_b \times \mu_o\,\vec{H})$ is a power interaction term between the beam current and the Lorentz type electric field introduced by the drifting beam.

(4) $\frac{m}{e}(\vec{V}_b \times \vec{J}_b^*) \cdot (\nabla \times \vec{v}_b)$ is a power interaction term which arises because of the rotational nature of the veam velocity, $\vec{v}_b$. This term is familiar in form to fluid dynamicists and for cases where $\vec{v}_b \times \nabla \times \vec{v}_b$ is zero for nonvanishing $\vec{v}_b$ and $\nabla \times \vec{v}_b$, the velocity vector is sometimes called the Beltrami vector and the flow is called Beltrami flow. Again, we point out the fact that the last two terms of (A-24) cancel if the collision frequency in the beam, ν_b , is zero.

The terms on the left side of (A-24) correspond to energy flux, where:

(1) $\vec{E} \times \vec{H}^*$ is the electromagnetic Poynting Vector,

(2) $P_p\,\vec{v}_p^*$ is the accoustic energy flux of the plasma waves,

(3) $\frac{-P_b\,\vec{J}_b^*}{N_b e}$ is the accoustic energy flux of the

beam waves,

(4) $\frac{-m}{e}\vec{J}_b^{*}(\vec{V}_b \cdot \vec{v}_b)$ is the d.c. to a.c. conversion term that accounts for the gain in the beam-plasma interaction. It is important to note that the a.c. velocity of the beam must have a component of velocity parallel to the d.c. beam velocity in order for this energy transfer to take place.

APPENDIX B

DERIVATION OF THE POTENTIAL FUNCTION EQUATIONS

We shall now derive a set of coupled second order differential equations involving potential functions from which the total field solution is obtained.

Taking the transverse curl of Equations (2-63) and (2-65), and using the relations (2-58)-(2-62), we obtain:

$$- \vec{a}_z \nabla_t^2 E_z - jk \vec{a}_z \nabla_t \cdot \vec{E}_t + j\omega \mu_o \nabla_t \times \vec{H}_t = - \nabla_t \times \vec{J}_{mt} , \tag{B-1}$$

$$- \vec{a}_z \nabla_t^2 H_z - jk \vec{a}_z \nabla_t \cdot \vec{H}_t - j\omega \varepsilon_o \nabla_t \times \vec{E}_t$$

$$+ e \nabla_t \times (N_p \vec{v}_{pt} + N_b \vec{v}_{bt}) = \nabla_t \times \vec{J}_{et} . \tag{B-2}$$

The transverse divergence of Equations (2-63) and (2-65) gives

$$jk \vec{a}_z \cdot (\nabla_t \times \vec{E}_t) + j\omega \mu_o \nabla_t \cdot \vec{H}_t = - \nabla_t \cdot \vec{J}_{mt} \tag{B-3}$$

$$jk \vec{a}_z \cdot (\nabla_t \times \vec{H}_t) - j\omega \varepsilon_o \nabla_t \cdot \vec{E}_t + e N_p \nabla_t \cdot \vec{v}_{pt}$$

$$+ e N_b \nabla_t \cdot \vec{v}_{bt} = \nabla_t \cdot \vec{J}_{et} \tag{B-4}$$

Substituting Equation (2-64) into (B-3) and dividing by $j\omega \mu_o$ gives:

$$\nabla_t \cdot \vec{H}_t - jk H_z = - \frac{\nabla_t \cdot \vec{J}_{mt}}{j\omega \mu_o} + \frac{k}{\omega\mu_o} J_{mz} , \tag{B-5}$$

which is just a form of Maxwell's constitutive relation

$$\nabla \cdot \mu_o \vec{H} = - \frac{\nabla \cdot \vec{J}_m}{j\omega} = \rho_m \quad ; \tag{B-6}$$

where ρ_m is an equivalent magnetic charge density. Multiplying Equation (2-71) by the factor $\frac{e}{mu_p^2}$ gives:

$$e\,N_p\,\nabla_t \cdot \vec{v}_{pt} = - \frac{j\omega\, e\, P_p}{mu_p^2} + jk\, e\, N_p\, v_{pz} + \frac{e}{mu_p^2} S_p \quad . \tag{B-7}$$

From Equation (2-68),

$$v_{pz} = \frac{-\,e\,E_z}{m(j\omega + \nu_p)} + \frac{jk\,P_p}{mN_p(j\omega + \nu_p)} + \frac{\left\{F_{pz} - \frac{P_p}{m\,u_p^2\,N_p} F_{poz}\right\}}{m\,N_p(j\omega + \nu_p)} \quad . \tag{B-8}$$

Combining Equations (B-7) and (B-8) yields:

$$e\,N_p\,\nabla_t \cdot \vec{v}_{pt} = - \frac{j\,e}{mu_p^2} \left(\omega - \frac{k^2\,u_p^2}{(\omega - j\nu_p)} \right) P_p$$

$$- \frac{N_p\,e^2\,k\,E_z}{m(j\omega - j\nu_p)} + \frac{e}{mu_p^2} S_p + \frac{k\,e\left\{F_{pz} - \frac{P_p\,F_{poz}}{m\,u_p^2\,N_p}\right\}}{m(\omega - j\nu_p)} \quad . \tag{B-9}$$

We can derive the term $eN_b\,\nabla_t \cdot \vec{v}_{bt}$ in a similar manner. Equation (2-72) is rewritten:

$$e\, N_b\, \nabla_t \cdot \vec{v}_{bt} = -\frac{j\, e}{mu_b^2}(\omega - k\, V_b)\, P_b + jk\, e\, N_b\, v_{bz} + \frac{e}{mu_b^2} S_b \quad . \tag{B-10}$$

From Equation (2-70),

$$v_{bz} = \frac{jk\, P_b}{m\, N_b(j\{\omega - k\, V_b\} + \nu_b)} - \frac{e\, E_z}{m(j\{\omega - k\, V_b\} + \nu_b)} + \frac{\left\{F_{bz} - \dfrac{P_b}{m\, u_b^2\, N_b} F_{boz}\right\}}{m\, N_b(j\{\omega - k\, V_b\} + \nu_b)} \quad . \tag{B-11}$$

Combining Equations (B-10) and (B-11) yields:

$$e\, N_b\, \nabla_t \cdot \vec{v}_{bt} = -\frac{j\, e}{mu_b^2}\left(\{\omega - k\, V_b\} - \frac{k^2\, u_b^2}{\{\omega - k\, V_b - j\nu_b\}}\right) P_b - \frac{N_b\, e^2\, k}{m(\omega - k\, V_b - j\nu_b)} E_z + \frac{e\, S_b}{m\, u_b^2} + \frac{k\, e\left\{F_{bz} - \dfrac{P_b}{m\, u_b^2\, N_b} F_{boz}\right\}}{m(\omega - k\, V_b - j\nu_b)} \quad . \tag{B-12}$$

Equation (2-66) can be solved for $\nabla_t \times \vec{H}_t$ when Equations (B-8) and (B-11) are used.

$$\nabla_t \times \vec{H}_t = \frac{-e\,k\,P_p\,\vec{a}_z}{m(\omega - j\,\nu_p)} - \frac{e}{m}\left\{\frac{k\,\vec{a}_z}{(\omega - k\,V_b - j\nu_b)} + \frac{\vec{V}_b}{u_b^2}\right\}P_b$$

$$+ j\varepsilon_o\left\{\omega - \frac{\omega_b^2}{(\omega - k\,V_b - j\nu_b)} - \frac{\omega_p^2}{(\omega - j\,\nu_p)}\right\}\vec{E}_z + \vec{J}_{ez}$$

$$+ j\,\frac{e}{m}\,\frac{\left\{\vec{F}_{pz} - \dfrac{P_p}{m\,u_p^2\,N_p}\vec{F}_{poz}\right\}}{(\omega - j\,\nu_p)} \tag{B-13}$$

$$+ j\,\frac{e}{m}\left\{\vec{F}_{bz} - \frac{P_b}{m\,u_b^2\,N_b}\vec{F}_{boz}\right\}\Big/(\omega - k\,V_b - j\,\nu_b) \quad ,$$

where $\omega_{b,p}^2 = \dfrac{N_{b,p}\,e^2}{m\varepsilon_o}$ are the squares of the beam and plasma frequencies. Taking, now, the transverse divergence of Equation (2-65) and using Equations (B-9), (B-12), and (B-13); the expression for $\nabla_t \cdot \vec{E}_t$ reduces to

$$\nabla_t \cdot \vec{E}_t - jk\,E_z = \frac{-e\,P_b}{m\,u_b^2\,\varepsilon_o} - \frac{e\,P_p}{m\,u_p^2\,\varepsilon_o} + \frac{\nabla_t \cdot \vec{J}_{et}}{-j\omega\,\varepsilon_o} + \frac{k}{\omega\varepsilon_o}J_{ez}$$

$$+ \frac{e}{mu_b^2(j\omega\varepsilon_o)}S_b + \frac{e}{mu_p^2(j\omega\varepsilon_o)}S_p + \frac{j\,ke}{m\omega\varepsilon_o}\,\frac{\left\{F_{pz} - \dfrac{P_p}{m\,u_p^2\,N_p}F_{poz}\right\}}{(\omega - j\,\nu_p)} \tag{B-14}$$

$$+ \frac{j\,k\,e}{m\,\omega\varepsilon_o}\,\frac{\left\{F_{bz} - \dfrac{P_b}{m\,u_b^2\,N_b}F_{boz}\right\}}{(\omega - k\,V_b - j\nu_b)} \quad ,$$

which is just Maxwell's constitutive relation

$$\nabla \cdot \vec{E} = \frac{\rho_e}{\varepsilon_o} \quad ; \tag{B-15}$$

where ρ_e is the electric charge density and here,

$$\rho_e = -\frac{e\,P_p}{m\,u_p^2} - \frac{e\,P_b}{m\,u_b^2} - \frac{\nabla \cdot \vec{J}_e}{j\omega} + \frac{e\,S_p}{m\,u_p^2(j\omega)} + \frac{e\,S_b}{m\,u_b^2(j\omega)} \tag{B-16}$$

$$+ \frac{jk\,e}{m\omega} \frac{\left\{F_{pz} - \frac{P_p}{m\,u_p^2\,N_p} F_{poz}\right\}}{(\omega - j\,\nu_p)} + \frac{jk\,e}{m\omega} \frac{\left\{F_{bz} - \frac{P_b}{m\,u_b^2\,N_b} F_{boz}\right\}}{(\omega - k\,V_b - j\nu_b)} ,$$

where the ratios $\frac{P_{b,p}}{mu_{b,p}^2} = n_{b,p}$ are the a.c. beam and plasma charge densities.

Taking the transverse curl of Equation (2-67) and substituting Equations (2-64) and (B-9) and combining terms gives:

$$N_p\,e\,\nabla_t \times \vec{v}_{pt} = \frac{\omega\mu_o\,\varepsilon_o\,\omega_p^2}{(\omega - j\nu_p)} \vec{H}_z - \frac{e\Omega}{(\omega - j\nu_p)\,mu_p^2} \left\{\omega - \frac{k^2\,u_p^2}{(\omega - j\nu_p)}\right\} P_p$$

$$+ \frac{jk\,\varepsilon_o\Omega\,\omega_p^2}{(\omega - j\nu_p)^2} \vec{E}_z - \frac{je}{m(\omega - j\nu_p)} \nabla_t \times \left(\vec{F}_{pt} - \frac{P_p}{m\,u_p^2\,N_p} \vec{F}_{pot}\right) \tag{B-17}$$

$$- \frac{j\varepsilon_o\,\omega_p^2\,\vec{J}_{mz}}{(\omega - j\,\nu_p)} - \frac{j\,\vec{\Omega}}{(\omega - j\nu_p)} \left\{\frac{e\,S_p}{m\,u_p^2} - \frac{j\,k\,e}{m(\omega - j\nu_p)} \left(F_{pz} - \frac{P_p}{m\,u_p^2\,N_p} F_{poz}\right)\right\} ,$$

where $\vec{\Omega} = \frac{e\vec{B}_o}{m}$ is the cyclotron frequency vector.

In a similar manner, $N_b e \nabla_t \times \vec{v}_{bt}$ can be calculated by taking the transverse curl of Equation (2-69) and using Equations (2-64), (B-5), and (B-12).

$$N_b e \nabla_t \times \vec{v}_{bt} = - \frac{\omega_b^2 \varepsilon_o}{(\omega - k V_b - j\nu_b)} \left\{ - \mu_o(\omega - k V_b)\vec{H}_z - \frac{jk \Omega \vec{E}_z}{(\omega - kV_b - j\nu_b)} \right.$$
$$\left. + \frac{\vec{\Omega}}{u_b^2 N_b e} \left[(\omega - k V_b) - \frac{k^2 u_b^2}{(\omega - k V_b - j \nu_b)} \right] P_b \right\} \tag{B-18}$$
$$- \frac{j e}{m(\omega - k V_b - j\nu_b)} \nabla_t \times \left\{ \vec{F}_{bt} - \frac{P_b \vec{F}_{bot}}{m u_b^2 N_b} \right\} - \frac{j\omega_b^2 \varepsilon_o \vec{J}_{mz}}{(\omega - kV_b - j\nu_b)}$$
$$- \frac{j \vec{\Omega}}{(\omega - kV_b - j\nu_b)} \left\{ \frac{e}{mu_b^2} S_b + \frac{ke}{m} \frac{\left(F_{bz} - \frac{P_b}{m u_b^2 N_b} F_{boz} \right)}{(\omega - k V_b - j\nu_b)} \right\}$$

We can now rewrite Equation (B-1) using the relations (B-13) and (B-14).

$$\nabla_t^2 E_z + \left\{k_o^2 \left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)}\right) - k^2\right\} E_z$$

$$+ \frac{jk\,e}{m\,\varepsilon_o}\left\{\frac{\omega\,\mu_o\,\varepsilon_o}{(\omega - j\nu_p)} - \frac{1}{u_p^2}\right\} P_p + \left\{\frac{jk\,e}{m\,\varepsilon_o}\left(\frac{\omega\,\mu_o\,\varepsilon_o}{(\omega - kV_b - j\nu_b)} - \frac{1}{u_b^2}\right)\right.$$

$$\left. + \frac{j\omega\,\mu_o e\,V_b}{m\,u_b^2}\right\} P_b$$

$$= \vec{a}_z \cdot \nabla_t \times \vec{J}_{mt} - \frac{ke}{mu_p^2\,\varepsilon_o} S_p - \frac{ke}{mu_b^2\,\varepsilon_o} S_b + \frac{k\,\nabla_t \cdot \vec{J}_{et}}{\omega\,\varepsilon_o} \tag{B-19}$$

$$- j\,\frac{k^2 - k_o^2}{\omega\varepsilon_o} J_{ez} + \frac{(k^2 - k_o^2)e}{m\,\omega\varepsilon_o}\,\frac{\left\{F_{pz} - \dfrac{P_p\,F_{poz}}{m\,u_p^2\,N_p}\right\}}{(\omega - j\nu_p)}$$

$$+ \frac{(k^2 - k_o^2)e}{m\,\omega\varepsilon_o}\,\frac{\left\{F_{bz} - \dfrac{P_b}{m\,u_b^2\,N_b} F_{boz}\right\}}{(\omega - k\,V_b - j\nu_b)} \quad ,$$

where $k_o^2 = \omega^2\,\mu_o\,\varepsilon_o$.

Similarly, Equation (B-2) can be rewritten with the aid of Equations (2-64), (B-5), (B-17), and (B-18).

$$\nabla_t^2 H_z + \left\{k_o^2\left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2(\omega - kV_b)}{\omega^2(\omega - kV_b - j\nu_b)}\right) - k^2\right\} H_z$$

$$- jk\,\varepsilon_o\Omega\left\{\frac{\omega_p^2}{(\omega - j\nu_p)^2} + \frac{\omega_b^2}{(\omega - kV_b - j\nu_b)^2}\right\} E_z + \frac{\left\{e\Omega\left(\omega - \dfrac{k^2\,u_p^2}{(\omega - j\nu_p)}\right)\right\} P_p}{(\omega - j\nu_p)\, m\, u_p^2}$$

$$+ e\Omega\,\frac{\left\{(\omega - kV_b) - \dfrac{k^2\,u_b^2}{(\omega - kV_b - j\nu_b)}\right\} P_b}{(\omega - k\,V_b - j\nu_b) m\, u_b^2} =$$

$$- \vec{a}_z \cdot \nabla_t \times \vec{J}_{et} + \frac{k}{\omega\mu_o}\nabla_t \cdot \vec{J}_{mt} \qquad \text{(B-20)}$$

$$- \frac{j}{\omega\mu_o}\left\{k^2 - k_o^2\left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - k\,V_b - j\nu_b)}\right)\right\} J_{mz}$$

$$- \frac{je}{m(\omega - j\nu_p)}\vec{a}_z \cdot \nabla_t \times \left\{\vec{F}_{pt} - \frac{P_p}{mu_p^2\, N_p}\vec{F}_{pot}\right\}$$

$$- \frac{je}{m(\omega - kV_b - j\nu_b)}\vec{a}_z \cdot \nabla_t \times \left\{\vec{F}_{bt} - \frac{P_b}{m\,u_b^2\, N_b}\vec{F}_{bot}\right\}$$

$$- \frac{j\,\Omega}{(\omega - j\nu_p)}\left\{\frac{e}{mu_p^2} S_p + \frac{ke}{m(\omega - j\nu_p)}\left(F_{pz} - \frac{P_p\, F_{poz}}{m\,u_p^2\, N_p}\right)\right\}$$

$$- \frac{j\,\Omega}{(\omega - kV_b - j\nu_b)}\left\{\frac{e}{mu_b^2} S_b + \frac{ke}{m(\omega - kV_b - j\nu_b)}\left(F_{bz} - \frac{P_b}{m\,u_b^2\, N_b} F_{boz}\right)\right\} .$$

Taking the transverse divergence of (2-67) and using Equations (B-9), (B-14), and (B-17) we obtain the differential equation for the plasma pressure.

$$\nabla_t^2 P_p + \frac{1}{u_p^2}\left\{\omega^2 - j\nu_p\omega - k^2 u_p^2 - \omega_p^2 - \frac{\Omega^2}{(\omega - j\nu_p)^2}(\omega^2 - j\nu_p\omega - k^2u_p^2)\right\} P_p$$

$$- \left\{\frac{\omega_p^2}{u_b^2}\right\} P_b + \frac{m\Omega}{e}\left\{\frac{\omega\mu_o\ \varepsilon_o\ \omega_p^2}{\omega - j\ \nu_p}\right\} H_z + \left\{\frac{jk\ \varepsilon_o\ \Omega^2\ \omega_p^2 \frac{m}{e}}{(\omega - j\ \nu_p)^2}\right\} E_z$$

$$= N_p\ e\left\{\frac{\nabla_t \cdot \vec{J}_{et}}{j\omega\ \varepsilon_o} - \frac{k\ J_{ez}}{\omega\varepsilon_o}\right\} + \frac{j\ \varepsilon_o\ \Omega\ \omega_p^2}{(\omega - j\nu_p)} \frac{m}{e} J_{mz}$$

$$+ j\ \frac{\omega_p^2}{u_b^2\ \omega} S_b + j\left\{\frac{\omega_p^2 - \omega(\omega - j\nu_p)}{u_p^2\ \omega}\right\} S_p \qquad \text{(B-21)}$$

$$- j\ \frac{k}{\omega(\omega - j\ \nu_p)}\left\{\omega_p^2 - \omega(\omega - j\ \nu_p)\right\}\left\{F_{pz} - \frac{P_p}{m\ u_p^2\ N_p} F_{poz}\right\}$$

$$- j\ \frac{k\omega_p^2}{\omega}\ \frac{\left\{F_{bz} - \frac{P_b}{m\ u_b^2\ N_b} F_{boz}\right\}}{(\omega - k\ V_b - j\ \nu_b)} + j\ \frac{\vec{\Omega}}{(\omega - j\nu_p)} \cdot \nabla_t \times \left(\vec{F}_{pt} - \frac{P_p}{m\ u_p^2\ N_p}\vec{F}_{pot}\right)$$

$$+ \nabla_t \cdot \left(\vec{F}_{pt} - \frac{P_p}{m\ u_p^2\ N_p}\vec{F}_{pot}\right)$$

The equation for the beam pressure is obtained by taking the transverse divergence of (2-69) and using the relations (B-12), (B-13), (B-14), and (B-18).

$$\nabla_t^2 P_b + \frac{1}{u_b^2}\Bigg\{(\omega - kV_b)(\omega - kV_b - j\nu_b) - k^2u_b^2 - \omega_b^2$$

$$- \Omega^2 \frac{[(\omega - kV_b)(\omega - kV_b - j\nu_b) - k^2u_b^2]}{(\omega - kV_b - j\nu_b)^2}$$

$$+ \omega_b^2 V_b \mu_o \varepsilon_o \left(\frac{k\,u_b^2}{(\omega - kV_b - j\nu_b)} + V_b\right)\Bigg\} P_b + \left\{\frac{\omega_b^2\, kV_b\, u_o\, \varepsilon_o}{(\omega - j\nu_p)} - \frac{\omega_b^2}{u_p^2}\right\} P_p$$

$$+ j\left\{\frac{k\Omega^2 \frac{m}{e}\, \omega_b^2\, \varepsilon_o}{(\omega - kV_b - j\nu_b)^2} - \omega_b^2 \frac{m}{e} V_b\, \mu_o\, \varepsilon_o^2 \omega\left(1 - \frac{\omega_p^2}{\omega(\omega - j\nu_p)} - \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)}\right)\right\} E_z$$

$$+ \frac{(\omega - kV_b)\omega_b^2 \mu_o \varepsilon_o \frac{m}{e}\Omega}{(\omega - kV_b - j\nu_b)} H_z = \nabla_t \cdot \left(\vec{F}_{bt} - \frac{P_b}{m\, u_b^2\, N_b}\vec{F}_{bot}\right)$$

$$+ \left\{[-j(\omega - kV_b) + \nu_b] + j\frac{\omega_b^2}{\omega} + j\frac{\Omega^2}{(\omega - kV_b - j\nu_b)}\right\}\frac{S_b}{u_b^2} \qquad \text{(B-22)}$$

$$+ j\frac{\omega_b^2}{\omega}\frac{S_p^2}{u_p^2} - jk\left\{1 + \frac{\omega_b^2}{\omega(\omega - kV_b - j\nu_b)} - \frac{\Omega^2}{(\omega - kV_b - j\nu_b)^2}\right\}$$

$$\left\{F_{bz} - \frac{P_b}{m\, u_b^2\, N_b}F_{boz}\right\} + N_b e\left\{\frac{\nabla_t \cdot \vec{J}_{et}}{j\omega\, \varepsilon_o} - \frac{k}{\omega\varepsilon_o}J_{ez}\right\}$$

$$+ j\frac{\omega_b^2\, \varepsilon_o \frac{m}{e}\Omega}{(\omega - kV_b - j\nu_b)} J_{mz} - j\frac{k\,\omega_b^2}{\omega(\omega - j\nu_p)}\left\{F_{pz} - \frac{P_p}{m\, u_p^2\, N_p}F_{poz}\right\}$$

$$+ N_b e\, V_b\, \mu_o\, J_{ez} + j\frac{\vec{\Omega}}{(\omega - kV_b - j\nu_b)} \cdot \nabla_t \times \left\{\vec{F}_{bt} - \frac{P_b}{m\, u_b^2\, N_b}\vec{F}_{bot}\right\}$$

In the preceding development, it was rather academic to assume sources in the problem since our investigation concerns itself with the investigation of the modes that propagate in the beam-plasma system. However, it is instructive to note the effects of coupling when sources are introduced. One would expect that the problem of investigating proper excitation for a particular wave would be rather difficult since the sources are coupled in nearly all of the potential equations.

APPENDIX C

BOUNDARY CONDITIONS

Beam-Plasma--Plasma Interface

In a physical problem where an electron beam interpenetrates a plasma, the transverse motion of the beam electrons in the traveling wave produces a rippled boundary effect of the beam which is shown in Fig. (C-1) for a particular instant of time.

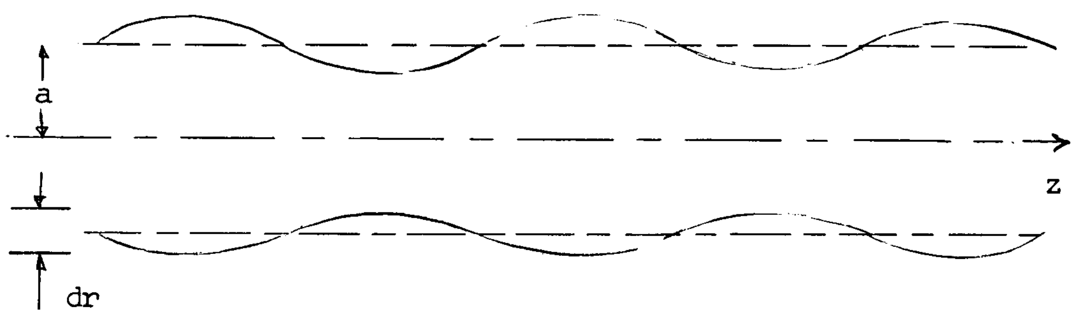

Fig. (C-1) Beam-Plasma--Plasma Interface

Since the boundary is not uniform at the nominal radius a for the beam, it would be difficult to apply boundary conditions at this interface.

Hahn[28] first used an artificial boundary condition which accounts for the rippling of the beam by considering the beam boundary to be fixed at the radius a and placing an equivalent

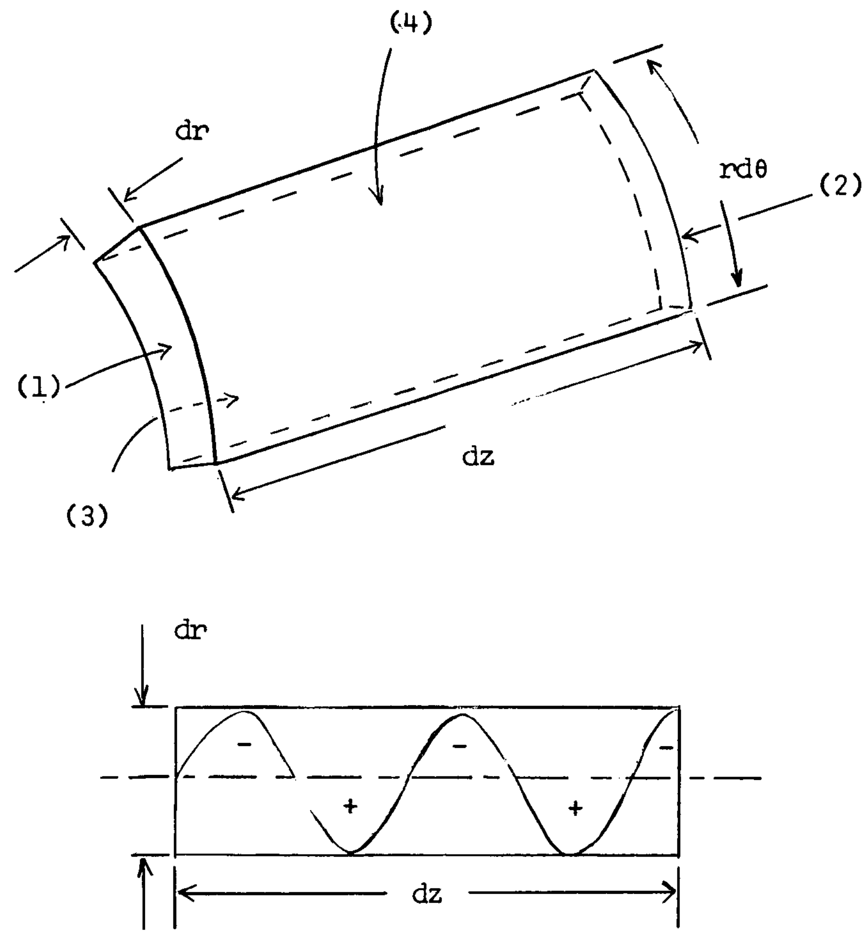

Fig. (C-2) Cylindrical Shell Volume ΔV

surface charge ρ_{sb} on the boundary to account for the net space charge of the scallop. The same approach will be used here to represent the scallop effect of the beam-plasma--plasma interface. We re-emphasize the fact that this is an artificial boundary condition that we are imposing on the beam but for situations where the rippling is very small, (dr small compared to a), this artificial boundary condition approaches the actual boundary condition.

For a cylindrical beam, we shall assume that we can construct a cylindrical shell around the beam-plasma--plasma interface that encloses the rippling of the beam as shown in Fig. (C-2).

Using the continuity equation for the beam

$$\nabla \cdot \left(N_b e \, \vec{v}_b + \frac{P_b \, e}{m \, u_b^2} \vec{V}_b \right) = - j\omega \frac{P_b \, e}{m \, u_b^2} \tag{C-1}$$

and the divergence theorem gives:

$$\int_{\Delta S} N_b e \, \vec{v}_b \cdot \vec{n} \, ds = \int_{\Delta V} (- j\omega + \vec{V}_b \cdot \nabla) \frac{P_b \, e}{m \, u_b^2} \, dV \tag{C-2}$$

which can be written as:

$$\int_{\Delta S} N_b e \, \vec{v}_b \cdot \vec{n} \, ds = - j(\omega - k \, V_b) \int_{\Delta V} \frac{P_b \, e}{m \, u_b^2} \, dV \tag{C-3}$$

The terms of the integrand of (C-3) are obtained from the first two terms of a Taylor series expansion and the surface integral terms for (1) and (2) become:

$$- N_b e\, v_{bz}\, rd\theta\, dr + \left(N_b e\, v_{bz} + \frac{\partial (N_b e\, v_{bz})}{\partial z}\, dz \right) rd\theta\, dr$$

$$= N_b e \frac{\partial v_{bz}}{\partial z}\, a\, d\theta\, dr\, dz \tag{C-4}$$

The integral over (3) and (4) gives:

$$- N_b e\, v_{br}\, rd\theta\, dz + 0 \tag{C-5}$$

where the term from (4) is zero since the beam velocity is zero on surface (4). From the remaining ends, we have the θ components which are given by:

$$\frac{\partial N_b e\, v_{b\theta}}{\partial \theta}\, dr\, dz\, d\theta = N_b e \frac{\partial v_{b\theta}}{\partial \theta}\, dr\, dz\, d\theta \quad . \tag{C-6}$$

The volume integral is given in the following form:

$$j(\omega - kV_b) \int_{\Delta V} \frac{e\, P_b}{m\, u_b^2}\, dV \approx j(\omega - kV_b) \frac{e\, P_b}{m\, u_b^2}\, a\, d\theta\, dr\, dz \quad . \tag{C-7}$$

Thus, Eq. (C-4) becomes, finally:

$$N_b e \frac{\partial v_{bz}}{\partial z}\, a\, d\theta\, dr\, dz - N_b e\, v_{br}\, a\, d\theta\, dz + N_b e \frac{\partial v_{b\theta}}{\partial \theta}\, dr\, dz\, d\theta$$

$$= j(\omega - kV_b) \frac{e\, P_b}{m\, u_b^2}\, a\, d\theta\, dr\, dz \tag{C-8}$$

Since the volume ΔV encloses the scallop, the term $\frac{e\, P_b}{m\, u_b^2}\, dr$ is finite and will be defined as the negative of the surface charge density, $-\rho_{sb}$. Therefore

$$j(\omega - k\, V_b)\, \rho_{sb} = N_b\, e\, v_{br} \tag{C-9}$$

since the other terms on the right side of Eq. (C-8) are on the order of dr smaller.

In the preceding development, it was assumed that dr was very small so that the contributions from the sides of the cylindrical shell were negligible compared to the contribution from the radial current. At the same time, dr must be large enough to enclose the total scallop of the beam. The thermal drift of the beam was neglected and the results obtained should be a good approximation to the actual boundary condition as long as the beam temperature is low so that the thermal drift is small during a period of the a.c. motion. Also, since dr is amplitude sensitive and must be large enough for ΔV to enclose the scallop, the boundary condition should be valid only for small signal analyses.

The question now arises whether one should also consider contribution to the boundary charge due to the plasma background in a manner similar to the development for the beam. It does not appear reasonable to include effects of the plasma in the equivalent surface charge since the reason for introducing the <u>artificial</u>

boundary condition was to account for the rippling of the beam! The charge distribution due to the plasma has already been accounted for by the term $-\frac{e\,P_p}{m\,u_p^2}$, which is the plasma volume charge density. One would certainly have to consider rippling in both beam and plasma if it were assumed that the beam and plasma occupied completely different regions of space as would be the case for an electron beam being surrounded by a plasma but not occupying the same region as the plasma.

Discontinuity in the Electric Field

At the interface of the beam-plasma--plasma system, the normal component of electric field is discontinuous by an amount proportional to the surface charge on the boundary. The tangential electric field must be continuous across the boundary. These conditions may be expressed in the following form:

$$\varepsilon_o\, \vec{n} \cdot \vec{E}\Big|_{a-}^{a+} = \rho_{sb} \qquad \text{(C-10)}$$

$$\vec{n} \times \vec{E}\Big|_{a-}^{a+} = 0 \qquad \text{(C-11)}$$

where $\vec{n}$ is the outward normal to the boundary surface, and (C-10) is a statement of Gauss's law for the interface.

Surface Current

The equivalent surface current at the beam-plasma--plasma interface is just equal to the equivalent beam surface charge density times the d.c. velocity of the beam. This surface current causes the azimuthal magnetic field to be discontinuous at the interface. Thus, the relations for the surface current and discontinuity in $\vec{H}$ become:

$$\vec{J}_s = \rho_{sb} \vec{V}_b \tag{C-12}$$

$$\vec{n} \times \vec{H}\Big|_{a-}^{a+} = \vec{J}_s \tag{C-13}$$

As a consequence of assuming that the plasma background occupies a region of space with the beam interpenetrating it and the fact that the beam and plasma are coupled only through the electric and magnetic fields, we must impose the additional boundary conditions that the plasma pressure P_p and the plasma velocity $\vec{v}_p$ are continuous across the interface. This is equivalent to specifying that the a.c. plasma density and a.c. plasma current be continuous across the interface.

We may now summarize the boundary conditions for the beam-plasma--plasma interface:

$$\frac{N_b e}{j(\omega - k V_b)} v_{br}\Big|_{a-} = \rho_{sb} \tag{C-14}$$

$$\varepsilon_o \vec{n} \cdot \vec{E}\Big|_{a-}^{a+} = \rho_{sb} \tag{C-15}$$

$$\vec{n} \times \vec{E}\Big|_{a-}^{a+} = 0 \tag{C-16}$$

$$\vec{n} \times \vec{H}\Big|_{a-}^{a+} = \vec{J}_s = \rho_{sb} \vec{V}_b \tag{C-17}$$

$$P_p\Big|_{a-}^{a+} = 0 \tag{C-18}$$

$$\vec{v}_p\Big|_{a-}^{a+} = 0 \quad . \tag{C-19}$$

APPENDIX D

UNIQUENESS OF POTENTIALS

Consider the set of equations

$$\nabla_t^2 \vec{U} + \{\lambda\} \vec{U} = 0 \tag{D-1}$$

where $\vec{U}$ is the vector of eigenfunctions and $\{\lambda\}$ is the diagonal eigenvalue matrix. The boundary conditions which these functions must satisfy are of the form:

$$\bar{\bar{\alpha}}\ \bar{\bar{S}}\ \vec{U} = \vec{f}$$

and (D-2)

$$\bar{\bar{\beta}}\ \bar{\bar{S}}\ \frac{\partial \vec{U}}{\partial n} = \vec{g}$$

on the boundary and $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ are of the form

$$\alpha \equiv \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{j1} & \cdots & \alpha_{jn} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ \beta_{(j+1)1} & \cdots & \beta_{(j+1)n} \\ \vdots & & \vdots \\ \beta_{n1} & & \beta_{nn} \end{bmatrix}$$

or a form where $\bar{\bar{\alpha}}$ and $\bar{\bar{\beta}}$ do not have terms in the same row.

To prove uniqueness, we shall assume that more than one eigenfunction satisfies each equation so that the difference solution satisfies the following equation:

$$\nabla_t^2 \vec{U}_D + \{\lambda\} \vec{U}_D = 0 \ . \qquad \text{(D-4)}$$

By premultiplying the conjugate of (D-4) by $\vec{U}_D$ and post multiplying (D-4) by $\vec{U}_D^*$ we arrive at the following relations when the two dimensional divergence theorem is used:

$$\int_C \vec{U}_D \frac{\partial \vec{U}_D}{\partial n} d\ell - \int_S \nabla_t \vec{U}_D \cdot \nabla_t \vec{U}_D^* dS + \int_S \{\lambda^*\} \vec{U}_D \vec{U}_D^* dS = 0 \quad \text{(D-5)}$$

$$\int_C \frac{\partial \vec{U}_D}{\partial n} \vec{U}_D^* d\ell - \int_S \nabla_t \vec{U}_D \cdot \nabla_t \vec{U}_D^* dS + \int_S \{\lambda\} \vec{U}_D \vec{U}_D^* dS = 0 \quad \text{(D-6)}$$

The form of the last terms in (D-5) and (D-6) were dependent on the fact that $\{\lambda\}$ is diagonal. On the contour C which encloses the surface S , the boundary conditions are:

$$\bar{\bar{\alpha}} \, \bar{\bar{S}} \, \vec{U}_D = 0$$

<u>and</u> $\qquad$ (D-7)

$$\bar{\bar{\beta}} \, \bar{\bar{S}} \, \frac{\partial \vec{U}_D}{\partial n} = 0$$

Thus, premultiplying (D-5) by $\bar{\bar{\alpha}} \, \bar{\bar{S}}$ and (D-6) by $\bar{\bar{\beta}} \, \bar{\bar{S}}$ and adding, we obtain

$$(\bar{\bar{\alpha}} + \bar{\bar{\beta}}) \, \bar{\bar{S}} \left\{ - \int_S \nabla_t \vec{U}_D \cdot \nabla_t \vec{U}_D^* dS + \{\lambda\} \int \vec{U}_D \vec{U}_D^* dS \right\}$$

$$- 2j \, \bar{\bar{\alpha}} \, \bar{\bar{S}} \, \mathrm{Im}(\{\lambda\}) \int_S \vec{U}_D \vec{U}_D^* dS = 0 \quad . \qquad \text{(D-8)}$$

Since $(\bar{\bar{\alpha}} + \bar{\bar{\beta}})$ and $\bar{\bar{S}}$ have inverse matrices, the relation becomes finally:

$$\left\{ - \int_S \nabla_t \vec{U}_D \cdot \nabla_t \vec{U}_D^{*} \, dS + \{\lambda\} \int_S \vec{U}_D \vec{U}_D^{*} \, dS \right\}$$

$$- 2j \, \mathrm{Im}(\{\lambda\}) \, \bar{\bar{S}}^{-1} (\bar{\bar{\alpha}} + \bar{\bar{\beta}})^{-1} \, \bar{\bar{\alpha}} \, \bar{\bar{S}} \int_S \vec{U}_D \vec{U}_D^{*} \, dS = 0 \qquad \text{(D-9)}$$

The first term in (D-9) has negative definite diagonal terms if $\vec{U}_D$ is not the zero vector. The diagonal parts of the second term are in general complex since $\{\lambda\}$ is generally complex. Thus, in order to have a non-trivial solution for the difference vector, the imaginary parts of the diagonals of the second and third terms at least must cancel. Thus,

$$\text{Diagonal Part} \left\{ \{\lambda''\} \, \mathrm{Re} \left[\left(\bar{\bar{I}} - 2\bar{\bar{S}}^{-1} (\bar{\bar{\alpha}} + \bar{\bar{\beta}})^{-1} \, \bar{\bar{\alpha}} \, \bar{\bar{S}} \right) \int_S \vec{U}_D \vec{U}_D^{*} \, dS \right] \right\} = 0 \qquad \text{(D-10)}$$

where λ'' is the imaginary part of $\{\lambda\}$ and $\bar{\bar{I}}$ is the unit matrix. Since λ'' is diagonal, (D-10) implies

$$\text{Diagonal Part} \left\{ \mathrm{Re} \left[\left(\bar{\bar{I}} - 2\bar{\bar{S}}^{-1} (\bar{\bar{\alpha}} + \bar{\bar{\beta}})^{-1} \, \bar{\bar{\alpha}} \, \bar{\bar{S}} \right) \int_S \vec{U}_D \vec{U}_D^{*} \, dS \right] \right\} = 0 \qquad \text{(D-11)}$$

For a given $\bar{\bar{\alpha}}, \bar{\bar{\beta}}$ one can check this condition and if any term is non zero, uniqueness is guaranteed. For boundary conditions of the form

$$\overline{\overline{\alpha}}\ \overline{\overline{S}}\ \vec{U} = \vec{f} \quad \text{on} \quad C$$

or (D-12)

$$\overline{\overline{\beta}}\ \overline{\overline{S}}\ \vec{U} = \vec{g} \quad \text{on} \quad C$$

where $\overline{\overline{\alpha}}$ or $\overline{\overline{\beta}}$ has an inverse, the uniqueness is easily seen from Equation (D-5) or (D-6).

APPENDIX E

COMPUTER PROGRAM

In this section, the computer program is given which was used to solve the bounded beam-plasma system. This problem was programmed in Algol 60 and run on an UNIVAC 1107 computer at Case Institute of Technology.

```
A GPS,* BPA,BPA
LISTED BY CASE 1107 GPS III (FAST VERSION) DATED JANUARY 20, 1967
THIS LISTING WAS DONE ON 20 APR 67 AT 00:48:08
      BEGIN
      REAL 2 ARRAY Y(1..2)  $
      REAL 2 ARRAY RLAM,ILAM(1..5)  $
      REAL 2 ARRAY RC,IC(1..5)  $
      REAL 2 NEWVAL,OLDVAL  $
      REAL 2 A,B,C,D,E,F,G,H,A2,B2,D2,E2,H2,C1,C2,C3,C4,C11,C13,C14,C17,C18,
      C21,C22,C23,R,VAL,DELTAB  $
      REAL 2 T  $
      REAL 2 A2H,A2T,AH,A2H2,AH2  $
      REAL 2 YSCALE,ISCALE  $
      INTEGER I  $
      INTEGER N  $
      BOOLEAN GATE  $
      FORMAT FORM1(E1,X2,'A=',R11.5,X3,'B=',R11.5,X3,'C=',R11.5,X3,
            'D=',R11.5,X3,'E=',R11.5,X3,'F=',R11.5,X3,'G=',R11.5,X3,
      'H=',R11.5,A0.1,'DELTAB=',R11.5,X3,'RADIUS(R)=',R11.5,X3,'YSCALE=',
      R11.5,X3,'ISCALE=',R11.5,A1)  $
      LIST INLIST(A,B,C,D,E,F,G,H,DELTAB,R,YSCALE,ISCALE)  $
      LOCAL LABEL NOMO  $

      PROCEDURE CXMULT(A,B,C,D,E,F)  $
      VALUE A,B,C,D  $
      REAL 2 A,B,C,D,E,F  $
      BEGIN
      E = A*C - B*D  $
      F = B*C + A*D  $
      END CXMULT  $

      PROCEDURE CXDIV(A,B,C,D,E,F)  $
      VALUE A,B,C,D  $
      REAL 2 A,B,C,D,E,F  $
      BEGIN
      REAL 2 G1  $
      REAL 2 C1,C2  $
      TEST.. IF (ABS(C) GTR &&+15) OR (ABS(D) GTR &&+15) THEN BEGIN
      A=A*&&-10  $  B=B*&&-10  $
      C=C*&&-10  $  D=D*&&-10  $
      GO TO TEST END  ELSE BEGIN
      G1= C*C + D*D  $  C1= A/G1  $  C2=  B/G1  $
      E=  C1*C + C2*D  $
      F= C2*C - C1*D  END
      END CXDIV  $

      REAL 2 PROCEDURE AV(RZ,IZ)  $
                                                             COMMENT
      THIS PROCEDURE COMPUTES THE ABSOLUTE VALUE OF THE COMPLEX NUMBER Z.  $
      VALUE RZ,IZ  $
      REAL 2 RZ,IZ  $
      BEGIN
      REAL 2 U  $
```

```
IF (IZ EQL 0.0 OR ABS(IZ) LSS &&-15*ABS(RZ)) THEN AV=ABS(RZ) ELSE BEGIN
U= RZ/IZ  $
IF (ABS(U) GTR 1.0) THEN AV=ABS(RZ)*SQRT(1.0+1.0/(U*U)) ELSE
AV=ABS(IZ)*SQRT(U*U + 1.0)  END  $
END AV  $

REAL 2 PROCEDURE INVTAN(N,D)  $
                                                          COMMENT
THIS IS AN INVERSE TANGENT PROCEDURE WHICH GIVES ANGLE FROM -PI TO PI $

VALUE N,D  $
REAL 2 N,D  $
BEGIN REAL 2 F,PI  $  F=0  $
PI=3.1415927  $
IF (D LSS 0.0) THEN BEGIN
IF (N LSS 0.0) THEN F=-PI ELSE F=PI  END  $
IF (D NEQ 0.0) THEN BEGIN
F=F+ARCTAN(N/D)  $  GOTO RET END  $
IF (N LSS 0.0) THEN BEGIN F=-PI/2  $
GO TO RET END  $
IF (N GTR 0.0) THEN BEGIN F=PI/2  $
GO TO RET END  $
F=0  $
RET..  INVTAN=F END INVTAN  $

PROCEDURE  NL(RZ,IZ,RNL,INL)  $
                                            COMMENT
THIS PROCEDURE COMPUTES THE NATURAL LOGARITHM OF THE COMPLEX NO. (Z)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RNL,INL  $  BEGIN
RNL=LN(AV(RZ,IZ))  $  INL=INVTAN(IZ,RZ)  $  END NL  $

PROCEDURE DETRM(RA,IA,RDET,IDET,NI)  $
                                                          COMMENT
THIS IS A DETERMINANT PROCEDURE FOR THE MATRIX A  $
VALUE RA,IA  $  REAL 2 ARRAY RA,IA  $  REAL 2 RDET,IDET  $ INTEGER NI  $
BEGIN
REAL 2 RW,IW,RD1,ID1,RD2,ID2,G  $
REAL 2 ARRAY RX,IX(1..NI,1..NI)  $
INTEGER I,J,K  $
FOR I=(1,1,NI) DO BEGIN FOR J=(1,1,NI) DO BEGIN
RX(I,J)=RA(I,J)  $  IX(I,J)=IA(I,J)  END END  $
FOR J=(1,1,NI) DO BEGIN
FOR I=(NI,-1,J+1) DO BEGIN
IF AV(RX(I,J),IX(I,J)) GTR AV(RX(I-1,J),IX(I-1,J)) THEN BEGIN
FOR K=(J,1,NI) DO BEGIN
RW=RX(I,K)  $  IW=IX(I,K)  $
RX(I,K)=RX(I-1,K)  $  IX(I,K)=IX(I-1,K)  $
RX(I-1,K)=-RW  $  IX(I-1,K)=-IW  END END END  $
FOR I=(J+1,1,NI) DO FOR K=(NI,-1,J) DO BEGIN
RD1=RX(J,K)*RX(I,J)-IX(J,K)*IX(I,J)  $
ID1=RX(J,K)*IX(I,J)+RX(I,J)*IX(J,K)  $
G=AV(RX(J,J),IX(J,J))  $
RD2=(RD1*RX(J,J)+ID1*IX(J,J))/G  $
ID2=(ID1*RX(J,J)-RD1*IX(J,J))/G  $
```

```
RD2=RD2/G  $  ID2=ID2/G  $
RX(I,K)=RX(I,K)-RD2  $  IX(I,K)=IX(I,K)-ID2 END END  $
RDET=1.0  $  IDET=0.0  $
FOR I=(1,1,NI) DO  CXMULT(RX(I,I),IX(I,I),RDET,IDET,RDET,IDET)  $
END DETRM  $

PROCEDURE COFAC(RA,IA,I,J,TEST,RLAM,ILAM,RCOF,ICOF,NR)  $
                                                                 COMMENT
IF TEST IS TRUE,PROCEDURE CALCULATES THE I,JTH COFACTOR OF A-LAMDA(I).
IF FALSE,THE I,JTH COFACTOR OF A.  $
VALUE I,J,TEST,RLAM,ILAM  $
REAL 2 ARRAY RA,IA  $  REAL 2 RLAM,ILAM,RCOF,ICOF  $  INTEGER I,J,NR  $
BOOLEAN TEST  $
BEGIN
INTEGER K,L,M,N  $  REAL 2 ARRAY RB,IB(1..NR-1,1..NR-1)  $
REAL 2 R1,I1  $
M=0  $
FOR K=(1,1,I-1),(I+1,1,NR) DO BEGIN  M=M+1  $    N=0  $
FOR L=(1,1,J-1),(J+1,1,NR) DO BEGIN N=N+1  $
IF K EQL L AND TEST THEN BEGIN
RB(M,N)=RA(K,L)-RLAM  $  IB(M,N)=IA(K,L)-ILAM  END  ELSE BEGIN
RB(M,N)=RA(K,L)  $  IB(M,N)=IA(K,L) END END END  $
DETRM(RB,IB,R1,I1,NR-1)  $
RCOF=((-1.0)**(I+J))*R1  $  ICOF=((-1.0)**(I+J))*I1  $
END COFAC  $

REAL 2 ARRAY PH(0..250)  $
INTEGER ARRAY FCT(0..25)  $
INTEGER ARRAY PR0(-1..17)  $
INTEGER ARRAY PR1(-1..17)  $

PROCEDURE P0(RZ,IZ,RP0,IP0)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RP0,IP0  $  BEGIN
REAL 2 RS,IS,R1,I1,R2,I2,R3,I3 $
CXDIV(1.0,0.0,64*(RZ*RZ-IZ*IZ),128*(RZ*IZ),RS,IS)  $
CXMULT(RS,IS,(-PR0(11)/FCT(6))+(PR0(15)/FCT(8))*RS,(PR0(15)
/FCT(8))*IS,R1,I1)  $
CXMULT(RS,IS,(PR0(7)/FCT(4))+R1,I1,R2,I2)  $
CXMULT(RS,IS,(-PR0(3)/FCT(2)) + R2,I2,R3,I3)  $
RP0=1.0 + R3  $  IP0=I3  $  END P0  $

PROCEDURE Q0(RZ,IZ,RQ0,IQ0)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RQ0,IQ0  $  BEGIN
REAL 2 RS,IS,R1,I1,R2,I2,R3,I3,R4,I4  $
CXDIV(1.0,0.0,64*(RZ*RZ-IZ*IZ),128*(RZ*IZ),RS,IS)  $
CXMULT(RS,IS,(PR0(13)/FCT(7))+(-PR0(17)/FCT(9))*RS,(-PR0(17)
/FCT(9))*IS,R1,I1)  $
CXMULT(RS,IS,(-PR0(9)/FCT(5))+R1,I1,R2,I2)  $
CXMULT(RS,IS,(PR0(5)/FCT(3))+R2,I2,R3,I3)  $
CXDIV(1.0,0.0,8*RZ,8*IZ,R4,I4)  $
CXMULT(R4,I4,-1.0+R3,I3,RQ0,IQ0)  $  END Q0  $

PROCEDURE P1(RZ,IZ,RP1,IP1)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RP1,IP1  $  BEGIN
```

```
REAL 2 RS,IS,R1,I1,R2,I2,R3,I3  $
CXDIV(1.0,0.0,64*(RZ*RZ-IZ*IZ),128*(RZ*IZ),RS,IS)  $
CXMULT(RS,IS,(-PR1(11)/FCT(6))+(PR1(15)/FCT(8))*RS,(PR1(15)
/FCT(8))*IS,R1,I1)  $
CXMULT(RS,IS,(PR1(7)/FCT(4))+R1,I1,R2,I2)  $
CXMULT(RS,IS,(-PR1(3)/FCT(2))+R2,I2,R3,I3)  $
RP1= 1.0 + R3  $  IP1= I3  $  END P1  $

PROCEDURE  Q1(RZ,IZ,RQ1,IQ1)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RQ1,IQ1  $  BEGIN
REAL 2 RS,IS,R1,I1,R2,I2,R3,I3,R4,I4  $
CXDIV(1.0,0.0,64*(RZ*RZ-IZ*IZ),128*(RZ*IZ), RS,IS)  $
CXMULT(RS,IS,(-PR1(13)/FCT(7))+(PR1(17)/FCT(9))*RS,(PR1(17)
/FCT(9))*IS,R1,I1)  $
CXMULT(RS,IS,(PR1(9)/FCT(5))+R1,I1,R2,I2)  $
CXMULT(RS,IS,(-PR1(5)/FCT(3))+R2,I2,R3,I3)  $
CXDIV(1.0,0.0,8*RZ,8*IZ,R4,I4)  $
CXMULT(R4,I4,3.0+R3,I3,RQ1,IQ1)  $     END Q1  $

PROCEDURE J0(RZ,IZ,RJ0,IJ0)  $
                                                            COMMENT
THIS PROCEDURE COMPUTES THE ZEROTH ORDER BESSEL FUNCTION OF ARGUMENT Z $
VALUE RZ,IZ  $
REAL 2 RZ,IZ,RJ0,IJ0  $
BEGIN
INTEGER J,M  $
REAL 2 RT,IT,RS,IS,R1,I1  $
RS = 0.0$ IS = 0.0$ M = 550$
FOR J = (0,1,M) DO BEGIN IF (J EQL 0) THEN BEGIN RT = 1.0$ IT = 0.0 END
ELSE BEGIN
CXMULT(-RZ/2.0,-IZ/2.0,RZ/2.0,IZ/2.0,R1,I1)  $
CXMULT(RT/(J*J),IT/(J*J),R1,I1,RT,IT)  END  $
IF AV(RS,IS)/AV(RT,IT) GTR &&+9 THEN M=J  $  RS=RS + RT  $
IS = IS + IT  END$ RJ0 = RS$ IJ0 = IS$
END J0  $

PROCEDURE J1(RZ,IZ,RJ1,IJ1)  $
                                                            COMMENT
THIS PROCEDURE COMPUTES THE FIRST ORDER BESSEL FUNCTION OF ARGUMENT Z $
VALUE RZ,IZ  $
REAL 2 RZ,IZ,RJ1,IJ1  $
BEGIN
INTEGER J,M  $
REAL 2 RT,IT,RS,IS,R1,I1  $
RS = 0.0$ IS = 0.0$ M = 550$
FOR J = (0,1,M) DO BEGIN IF (J EQL 0) THEN BEGIN
RT = RZ/2.0 $ IT = IZ/2.0 END ELSE BEGIN
CXMULT(-RZ/2.0,-IZ/2.0,RZ/2.0,IZ/2.0,R1,I1)  $
CXMULT(RT/(J*(J+1.0)),IT/(J*(J+1.0)),R1,I1,RT,IT)  END  $
IF AV(RS,IS)/AV(RT,IT) GTR &&+9 THEN M=J  $  RS=RS + RT  $
IS=IS+IT  $
END$
RJ1 = RS$  IJ1 = IS$
END J1  $
```

```
PROCEDURE Y0(RZ,IZ,RY0,IY0) $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RY0,IY0  $     BEGIN
INTEGER M,J  $  REAL 2 RS,IS,RT,IT,R1,I1,R2,I2,R3,RNL,INL,RJ0,IJ0  $
M=550  $
J0(RZ,IZ,RJ0,IJ0)  $  NL(RZ/2,IZ/2,RNL,INL)  $
CXMULT((RJ0/1.5707963),(IJ0/1.5707963),(0.5772157+RNL),INL,RS,IS)  $
FOR J=(1,1,M) DO BEGIN IF J EQL 1 THEN BEGIN
CXMULT(RZ/4,IZ/4,(RZ/1.5707963),(IZ/1.5707963),RT,IT)  $  END ELSE
BEGIN
CXMULT(-(RZ/2),-(IZ/2),RZ/2,IZ/2,R1,I1)  $
R3=PH(J)/(J*J*PH(J-1))  $
CXMULT(RT,IT,R1*R3,I1*R3,RT,IT)  $  END  $
IF AV(RS,IS)/AV(RT,IT)  GTR &&+9  THEN M=J  $
RS=RS+RT  $  IS=IS+IT  $  END  $  RY0=RS  $  IY0=IS  $  END  Y0  $

PROCEDURE Y1(RZ,IZ,RY1,IY1)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RY1,IY1  $  BEGIN
INTEGER M,J  $  REAL 2 RS,IS,RT,IT,R1,I1,R2,I2,R3,I3,R4,I4,RJ1,IJ1,RNL,
INL  $  M=550  $
J1(RZ,IZ,RJ1,IJ1)  $  NL(RZ/2,IZ/2,RNL,INL)  $
CXDIV(1.0,0.0,(1.5707963*RZ),(1.5707963*IZ),R1,I1)  $
CXMULT((RJ1/1.5707963),(IJ1/1.5707963),(0.5772157+RNL),INL,R2,I2)  $
RS=R2-R1-(RZ/(6.2831853))  $  IS=I2-I1-(IZ/(6.2831853))  $
FOR J=(1,1,M) DO BEGIN IF J EQL 1 THEN BEGIN
CXMULT(RZ/2,IZ/2,RZ/2,IZ/2,R3,I3)  $
CXMULT((R3*(1.25/3.1415926)),(I3*(1.25/3.1415926)),RZ/2,IZ/2,RT,IT)  $
END ELSE BEGIN R4=(PH(J)+PH(J+1))/(J*(J+1)*(PH(J-1)+PH(J)))  $
CXMULT(RT,IT,(-R4*R3),(-R4*I3),RT,IT)  $  END $
IF AV(RS,IS)/AV(RT,IT)  GTR &&+9 THEN M=J  $
RS=RS+RT  $  IS= IS+IT  $  END  $  RY1=RS  $  IY1=IS  $  END Y1  $

PROCEDURE  JJ(RZ,IZ,RJJ,IJJ)  $
VALUE RZ,IZ  $  REAL 2 RZ,IZ,RJJ,IJJ$  BEGIN
REAL 2 RJ0,IJ0,RJ1,IJ1,RP0,IP0,RP1,IP1,RQ0,IQ0,RQ1,IQ1  $
IF AV(RZ,IZ) LEQ 10 THEN BEGIN
J0(RZ,IZ,RJ0,IJ0)  $  J1(RZ,IZ,RJ1,IJ1)  $
CXDIV(RJ0,IJ0,RJ1,IJ1,RJJ,IJJ) END
ELSE BEGIN
P0(RZ,IZ,RP0,IP0)  $  Q0(RZ,IZ,RQ0,IQ0)  $
P1(RZ,IZ,RP1,IP1)  $  Q1(RZ,IZ,RQ1,IQ1)  $
CXDIV(IP0-RQ0,(-RP0-IQ0),RP1+IQ1,IP1-RQ1,RJJ,IJJ)  END END JJ  $

PROCEDURE HH(RZ,IZ,RHH,IHH)  $
VALUE RZ,IZ $ REAL 2 RZ,IZ,RHH,IHH $  BEGIN
REAL 2 RJ0,IJ0,RY0,IY0,RJ1,IJ1,RY1,IY1  $
REAL 2 RP0,IP0,RP1,IP1,RQ0,IQ0,RQ1,IQ1  $
IF AV(RZ,IZ) LEQ 4 THEN BEGIN
J0(RZ,IZ,RJ0,IJ0)  $  J1(RZ,IZ,RJ1,IJ1)  $
Y0(RZ,IZ,RY0,IY0)  $  Y1(RZ,IZ,RY1,IY1)  $
CXDIV(RJ0-IY0,IJ0+RY0,RJ1-IY1,IJ1+RY1,RHH,IHH)  $  END ELSE
BEGIN
P0(RZ,IZ,RP0,IP0)  $  P1(RZ,IZ,RP1,IP1)  $
Q0(RZ,IZ,RQ0,IQ0)  $  Q1(RZ,IZ,RQ1,IQ1)  $
```

```
CXDIV(-(RQ0+IP0),RP0-IQ0,RP1-IQ1,IP1+RQ1,RHH,IHH)$ END END HH  $

PROCEDURE POLROOT (RH,IH) DEGREE:(HN) ROOTS:(RZ,IZ)  $
                                                            COMMENT
AFTER EACH  ROOT IS FOUND, IT IS SUBSTITUTED BACK INTO THE POLYNOMIAL
THIS PROCEDURE CALCULATES THE N COMPLEX ROOTS OF AN  N TH DEGREE
POLYNOMIAL WITH COMPLEX COEFFICIENTS. IT USES A COMPLEX NEWTON'S METHOD.
WHICH IS THEN REDUCED BY ONE IN DEGREE AND THE NEXT ROOT IS CALCULATED.

RH AND IH ARE THE THE COEFFICIENTS OF THE POLYNOMIAL IN Z AND HN IS
EQUAL TO THE DEGREE PLUS ONE.  $
VALUE RH,IH,HN  $  REAL 2 ARRAY RH,IH,RZ,IZ  $  INTEGER HN  $
BEGIN  INTEGER CV,NCV,M,TP,PP,N,L,K  $
INTEGER ARRAY ORDER(1..HN)  $
INTEGER ARRAY CODE(1..HN)  $
INTEGER COUNT,I,J,S  $
REAL 2 ARRAY RB,IB,RC,IC,RZF,IZF,RF,IE(1..HN)  $
REAL 2 EPS,R1,I1,R2,I2,RZP,IZP,E,F  $
REAL 2 SM  $
BOOLEAN CMPLX  $
LOCAL LABEL HOME  $
LOCAL LABEL ACCURATE  $
                                                         COMMENT
DETERMINE IF THE COEFFICIENTS OF THE POLYNOMIAL IN Z ARE COMPLEX  $
SM= 0.0  $
FOR I=(1,1,HN) DO SM=SM+ABS(IH(I))  $
CMPLX=(SM NEQ 0.0)  $

TP=HN+1  $  PP=HN  $
FOR N = (1,1,HN) DO BEGIN  RF(N) = RH(N)  $  IE(N) = IH(N) ENDS
RB(1)=RC(1)=1.0  $
IB(1)=IC(1)= 0.0  $
                                                         COMMENT
FIND THE NEXT ROOT.  $
REP..TP=TP-1  $  PP=PP-1  $  M=M+1  $RZ(M)=IZ(M)=0.5  $
COUNT=0  $  EPS=&&-6  $
ITER..
                                                         COMMENT
CALCULATE Z=Z-H(N)/H'(N) AS IN NEWTON'S METHOD.  $

COUNT=COUNT+1  $
FOR N=(2,1,TP) DO
    BEGIN L=N-1  $
R1=RZ(M)*RB(L)-IZ(M)*IB(L)  $  I1=RZ(M)*IB(L)+IZ(M)*RB(L)  $
RB(N)=R1+RH(N)  $  IB(N)=I1+IH(N)  $
R2=RZ(M)*RC(L)-IZ(M)*IC(L)  $  I2=RZ(M)*IC(L)+IZ(M)*RC(L)  $
RC(N)=R2+RB(N)  $  IC(N)=I2+IB(N)  $
    END  $

IF ((&&-20)*RC(PP)*RC(PP)+(&&-20)*IC(PP)*IC(PP)) LSS (&&-20) THEN BEGIN
RZP=(RB(TP)*RC(PP)+IB(TP)*IC(PP))/(RC(PP)*RC(PP)+IC(PP)*IC(PP))  $
IZP=(IB(TP)*RC(PP)-RB(TP)*IC(PP))/(RC(PP)*RC(PP)+IC(PP)*IC(PP)) END
ELSE BEGIN
RZP=((&&-20)*RB(TP)*RC(PP)+(&&-20)*IB(TP)*IC(PP))/
```

```
((&&-20)*RC(PP)*RC(PP)+(&&-20)*IC(PP)*IC(PP))  $
IZP=((&&-20)*IB(TP)*RC(PP)-(&&-20)*RB(TP)*IC(PP))/
((&&-20)*RC(PP)*RC(PP)+(&&-20)*IC(PP)*IC(PP)) END  $
RZ(M)=RZ(M)-RZP  $  IZ(M)=IZ(M)-IZP  $
                                                                  COMMENT
TEST FOR CONVERGENCE OF ROOT.  $

IF (SQRT(RZP*RZP+IZP*IZP) GTR EPS) AND (COUNT LSS 60) THEN GO TO ITER $
                                                                  COMMENT
ROOT CONVERGED.  IS THIS THE LAST ROOT.  IF NOT THEN SUBSTITUTE THIS
ROOT BACK INTO POLYNOMIAL, REDUCE THE DEGREE BY ONE, AND GO BACK TO FIND
THE NEXT ROOT.   $

IF TP LEQ 2 THEN GO TO ACCURATE  $
FOR N=(1,1,TP) DO
    BEGIN RH(N)=RB(N)  $  IH(N)=IB(N)
    END  $
GO TO REP  $
ACCURATE..
EPS=&&-8  $  M=0  $  TP=HN  $  PP=HN-1  $  S=1  $  J=1  $
IF CMPLX THEN GO TO ALT  $
AGAIN..
IZF(J)=-1  $  FOR I=(1,1,HN-1) DO
BEGIN IF IZ(I) GTR IZF(J) THEN
   BEGIN IZF(J)=IZ(I)  $  RZF(J)=RZ(I)  $K=I END END  $
IZ(K)=-1.0  $
IF ABS(IZF(J)) GTR &&-3 THEN BEGIN CODE(J)=2  $  S=S+2  END
ELSE BEGIN CODE(J)=1  $  S=S+1 END  $
IF S LSS HN THEN BEGIN J=J+1  $  GO TO  AGAIN END  $
IF S GTR HN THEN BEGIN WRITE('ERROR IN ACCURATE ROUTINE - ANSWERS ARE NO
T CORRECT')  $  GO TO HOME END  $
GO TO PLUGIN  $
ALT.. FOR J=(1,1,HN-1) DO BEGIN
I=1  $
   A1.. IF (ORDER(I) EQL 0) THEN BEGIN
   RZF(J)=RZ(I)  $  IZF(J)=IZ(I)  END
   ELSE BEGIN I=I+1  $ GO TO A1 END  $
K=L=I  $
FOR I=(K+1,1,HN-1) DO BEGIN
IF (RZF(J) LSS RZ(I)) AND (ORDER(I) EQL 0) THEN BEGIN
   RZF(J)=RZ(I)  $  IZF(J)=IZ(I)  $
   L=I END END  $
ORDER(L)=1 END  $
FOR I=(1,1,HN-1) DO BEGIN
RZ(I)=RZF(I)  $  IZ(I)=IZF(I)  $  ORDER(I)=0  END  $
FOR J=(1,1,HN-1) DO CODE(J)=1  $
J=HN-1  $
PLUGIN.. K=J  $
FOR J=(1,1,K) DO BEGIN COUNT=0  $  EPS=&&-8  $
M=M+1  $  RZ(M)=RZF(J)  $  IZ(M)=IZF(J)  $
ITERF..
COUNT=COUNT+1  $
FOR N=(2,1,TP) DO
BEGIN L=N-1  $
```

```
R1=RZ(M)*RB(L)-IZ(M)*IB(L)  $
I1=RZ(M)*IB(L)+IZ(M)*RB(L)  $
RB(N)=R1+RF(N)  $  IB(N)=I1+IE(N)  $
R2=RZ(M)*RC(L)-IZ(M)*IC(L)  $
I2=RZ(M)*IC(L)+IZ(M)*RC(L)  $
RC(N)=R2+RB(N)  $  IC(N)=I2+IB(N) END  $
RZP=(RB(TP)*RC(PP)+IB(TP)*IC(PP))/(RC(PP)*RC(PP)+IC(PP)*IC(PP))  $
IZP=(IB(TP)*RC(PP)-RB(TP)*IC(PP))/(RC(PP)*RC(PP)+IC(PP)*IC(PP))  $
RZ(M)=RZ(M)-RZP  $  IZ(M)=IZ(M)-IZP  $
IF (SQRT(RZP*RZP+IZP*IZP) GTR EPS) AND (COUNT LSS 60) THEN GO TO ITERF $
IF CODE(J) EQL 2 THEN
BEGIN M=M+1  $
RZ(M)=RZ(M-1)  $
IZ(M)=-IZ(M-1) END END  $
HOME..END POLEROOT  $

REAL 2 PROCEDURE FUNCTION(X)  $
                                                          COMMENT
CALCULATE THE OBJECTIVE BOUNDARY DETERMINANT WHICH IF MINIMIZED HAS THE
VALUE OF ZERO.  $
VALUE X  $  REAL 2 ARRAY X  $
BEGIN
REAL 2 RK,IK,RK2,IK2,RN,IN,RM,IM,RO,IO,RL,IL,RP,IP,RKL,IKL,RKOM,IKOM,
RMN,IMN,ARG,MAG,RDET,IDET,X1,X2,X3,X4,X5,X6,X7,X8,
X9,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9  $
REAL 2 R1,R2,R3,R4,R5,R6,I1,I2,I3,I4,I5,I6  $
REAL 2 RL1,RL2,RL3,RL4,RL5,RL6,RL7,RL8,RL9,RL10,RL11,RL12,RL13,
       IL1,IL2,IL3,IL4,IL5,IL6,IL7,IL8,IL9,IL10,IL11,IL12,IL13,
RXL,IXL,RAPD,IAPD,RAMD,IAMD,RAM,IAM,RAP,IAP,RPM,IPM,RPN,IPN  $
REAL 2 ARRAY RAR,IAR,RS,IS(1..3,1..3),RA,IA(1..4),RJJ,IJJ(1..3),
RHH,IHH(4..5),RT,IT(1..2,4..5),RH,IH(3..5,1..2),RD,ID,RG,IG(1..5,1..5) $
REAL 2 ARRAY RLAMP,ILAMP(1..3)  $
INTEGER I,J,K  $
INTEGER L  $  INTEGER ARRAY ORDER(1..3)  $
RK=X(2)  $  IK=ISCALE*X(1)  $
RK2 = RK*RK - IK*IK  $
IK2 = 2.0*RK*IK  $
RN = A - (RK*C)/H  $
IN = -(IK*C)/H  $
RM = RN  $
IM = IN - B*G  $
RO = A2 - RK2/C11  $
IO = - A*F - IK2/C11  $
CXDIV(1.0,0.0,A,-F,RL,IL)  $
CXDIV(B2,0.0,RM,IM,RP,IP)  $
CXMULT(RK,IK,RL,IL,RKL,IKL)  $
CXDIV(RK,IK,RM,IM,RKOM,IKOM)  $
CXMULT(RM,IM,RN,IN,RMN,IMN)  $
CXMULT(RP,IP,RN,IN,RPN,IPN)  $
RAR(1,1)=C1*(1.0-(RL+RP)/A)-RK2  $
IAR(1,1)=-C1*((IL+IP)/A)-IK2  $
RAR(1,2)=-C2*IKL+C3*IK  $
IAR(1,2)=C2*RKL-C3*RK  $
RAR(1,3)=-C2*IKOM+C4*IK  $
```

```
IAR(1,3)=C2*RKOM-C4*RK+C22  $
RAR(2,1)=0.0  $
IAR(2,1)=0.0  $
RAR(2,2)=C11*RO-C11  $
IAR(2,2)=C11*IO  $
RAR(2,3)=-C13  $
IAR(2,3)=0.0  $
RAR(3,1)=-C14*((IL+IP)/A)  $
IAR(3,1)=-C14*(1.0-(RL+RP)/A)  $
RAR(3,2)=C17*RKL-C18  $
IAR(3,2)=C17*IKL  $
RAR(3,3)=C13*RMN-RK2-C23+C17*RKOM+C21  $
IAR(3,3)=C13*IMN-IK2+C17*IKOM  $

                                                                  COMMENT
CALCULATE THE COEFFICIENTS OF THE CHARACTERISTIC POLYNOMIAL.  $
RA(1)=1.0  $  IA(1)=0.0  $
RA(2)=-(RAR(1,1)+RAR(2,2)+RAR(3,3))  $
IA(2)=-(IAR(1,1)+IAR(2,2)+IAR(3,3))  $
CXMULT(RAR(1,1),IAR(1,1),RAR(2,2)+RAR(3,3),IAR(2,2)+IAR(3,3),X1,Y1)  $
CXMULT(RAR(2,2),IAR(2,2),RAR(3,3),IAR(3,3),X2,Y2)  $
CXMULT(RAR(1,3),IAR(1,3),RAR(3,1),IAR(3,1),X3,Y3)  $
CXMULT(RAR(2,3),IAR(2,3),RAR(3,2),IAR(3,2),X4,Y4)  $
CXMULT(RAR(1,1),IAR(1,1),X4-X2,Y4-Y2,X5,Y5)  $
CXMULT(RAR(1,2),IAR(1,2),RAR(2,3),IAR(2,3),X6,Y6)  $
CXMULT(RAR(3,1),IAR(3,1),X6,Y6,X7,Y7)  $
CXMULT(RAR(1,3),IAR(1,3),RAR(2,2),IAR(2,2),X8,Y8)  $
CXMULT(RAR(3,1),IAR(3,1),X8,Y8,X9,Y9)  $
RA(3)= X1+X2-X3-X4  $
IA(3)= Y1+Y2-Y3-Y4  $
RA(4)= X5-X7+X9  $
IA(4)= Y5-Y7+Y9  $
                                                                  COMMENT
CALCULATE THE EIGENVALUES(LAMDA SQUARED) OF THE MATRIX AR.  $
POLROOT(RA,IA,4,RLAM,ILAM)  $

                                                                  COMMENT              2
CALCULATE THE SIMILARITY TRANSFORM MATRIX REALIZING THAT AR(2,1)=0.  $                 2
CXMULT(RAR(2,3),IAR(2,3),RAR(3,1),IAR(3,1),R1,I1)  $                                   2
R2=RAR(2,2)+RAR(3,3)  $  I2=IAR(2,2)+IAR(3,3)  $                                       2
FOR I=(1,1,3) DO BEGIN                                                                 2
CXMULT(RLAM(I),ILAM(I),R2,I2,R3,I3)  $                                                 2
CXMULT(RLAM(I),ILAM(I),RLAM(I),ILAM(I),R4,I4)  $                                       2
CXDIV(X2-X4-R3+R4,Y2-Y4-I3+I4,R1,I1,RS(1,I),IS(1,I))  $                                2
RS(2,I)=1.0  $  IS(2,I)=0.0  $                                                         2
CXDIV(-RAR(2,2)+RLAM(I),-IAR(2,2)+ILAM(I),RAR(2,3),IAR(2,3),RS(3,I),                   2
IS(3,I))  END  $                                                                       2
                                                                  COMMENT
THE SQUARE ROOT OF THE EIGENVALUES GIVE THE TRANSVERSE WAVE NUMBERS.  $
RLAM(4)=C1*(1.0-RL/A)-RK2  $  ILAM(4)=-C1*(IL/A)-IK2  $
RLAM(5)=C11*(RO-1.0)  $  ILAM(5)=C11*IO  $
FOR I=(1,1,5) DO BEGIN
MAG = (RLAM(I)*RLAM(I) + ILAM(I)*ILAM(I))**(0.25)  $
ARG = INVTAN(ILAM(I),RLAM(I))/2.0  $
```

```
RLAM(I) = MAG*COS(ARG)  $
ILAM(I) = MAG*SIN(ARG) END  $
                                                          COMMENT
OBTAIN THE PROPER BRANCH FOR LAMDA(4) AND LAMDA(5) IN THE HH PROCEDURE $
FOR I=(4,1,5) DO BEGIN  IF ILAM(I) LSS 0.0 THEN BEGIN
ILAM(I)=-ILAM(I)  $  RLAM(I)=-RLAM(I)  END END  $
                                                          COMMENT
CALCULATE THE OBJECTIVE BOUNDARY VALUE MATRIX D(I,J)  $
RAPD = (RK2/A2H2) -1.0 + RL/A  $  IAPD = (IK2/A2H2) + IL/A  $
CXDIV(1.0,0.0,RAPD,IAPD,RAP,IAP)  $
RAMD = RAPD + RPN/A2  $  IAMD = IAPD + IPN/A2  $
CXDIV(1.0,0.0,RAMD,IAMD,RAM,IAM)  $
RXL=RK-A*C*H  $  IXL=IK  $  RL1=RP*C*H-RK  $  IL1=IP*C*H-IK  $
CXMULT(RAM,IAM,RL,IL,RL2,IL2)  $  CXMULT(RAP,IAP,RK,IK,RL3,IL3)  $
CXMULT(RAP,IAP,RL,IL,RL4,IL4)  $  CXMULT(RAM,IAM,RN,IN,RL5,IL5)  $
RL6=1.0 + RP/A  $  IL6=IP/A  $
CXMULT(RAM,IAM,RXL,IXL,RL7,IL7)  $  CXMULT(RAP,IAP,RXL,IXL,RL8,IL8)  $
CXMULT(RL5,IL5,RL6,IL6,RL9,IL9)  $  CXMULT(RL2,IL2,RL6,IL6,RL10,IL10)  $
CXMULT(-(IAM/A2T),RAM/A2T,RL1,IL1,RG(3,1),IG(3,1))  $
RG(3,2)=-1.0 + RL2/A  $  IG(3,2)=IL2/A  $
CXDIV(RAM,IAM,RM*A,IM*A,RG(3,3),IG(3,3))  $
RG(3,4)=-IL3/A2T  $  IG(3,4)=RL3/A2T  $
RG(3,5)=1.0 - RL4/A  $  IG(3,5)=-IL4/A  $
CXMULT(RL9,IL9,RL1/A2H,IL1/A2H,RL11,IL11)  $
RG(4,1)=IP*(C/A) -IL11  $  IG(4,1)=-RP*(C/A) + RL11  $
CXMULT(RN*T,IN*T,RL10/AH,IL10/AH,RG(4,2),IG(4,2))  $
CXDIV(-T*(1.0-RL9/A),T*(IL9/A),RM*H,IM*H,RG(4,3),IG(4,3))  $
CXMULT(-IL3,RL3,RN/A2H,IN/A2H,RG(4,4),IG(4,4))  $
CXMULT(-T*RN,-T*IN,RL4/AH,IL4/AH,RG(4,5),IG(4,5))  $
CXMULT(RL7,IL7,RL1/A2H2,IL1/A2H2,RL12,IL12)  $
RG(5,1)=-(1.0 + RL12)  $  IG(5,1)=-IL12  $
CXMULT(-T*IL7,T*RL7,RL/AH2,IL/AH2,RG(5,2),IG(5,2))  $
CXDIV(-T*IL7,T*RL7,RM*AH2,IM*AH2,RG(5,3),IG(5,3))  $
CXMULT(RL3,IL3,RXL/A2H2,IXL/A2H2,RL13,IL13)  $
RG(5,4)=1.0-RL13  $  IG(5,4)=-IL13  $
CXMULT(T*IL4,-T*RL4,RXL/AH2,IXL/AH2,RG(5,5),IG(5,5))  $
FOR I=(3,1,5) DO  FOR J=(1,1,2) DO BEGIN
RH(I,J)=-RG(I,J+3)  $  IH(I,J)=-IG(I,J+3) END  $
CXDIV(-IK*T,RK*T,H2*(A2-1.0),H2*(-A*F),RT(1,5),IT(1,5))  $
RT(1,4)=1.0  $  RT(2,5)=-1.0  $  IT(2,4)=IT(2,5)=IT(1,4)=RT(2,4)=0.0  $
FOR I=(1,1,2) DO BEGIN  FOR J=(1,1,3) DO BEGIN
RD(I,J)=RS(I,J)  $  ID(I,J)=IS(I,J)  END  $
FOR J=(4,1,5) DO BEGIN
RD(I,J)=-RT(I,J)  $  ID(I,J)=-IT(I,J) END END  $
FOR J=(1,1,3) DO  JJ(RLAM(J)*R,ILAM(J)*R,RJJ(J),IJJ(J))  $               7
FOR J=(4,1,5) DO HH(RLAM(J)*R,ILAM(J)*R,RHH(J),IHH(J))  $                 7
FOR I=(3,1,5) DO BEGIN  FOR J=(1,1,3) DO BEGIN  RD(I,J)=ID(I,J)=0.0  $    7
FOR K=(1,1,3) DO BEGIN                                                    7
CXMULT(RG(I,K),IG(I,K),RS(K,J),IS(K,J),R5,I5)  $                          7
RD(I,J)=RD(I,J) + R5  $  ID(I,J)=ID(I,J) + I5  END END  $                 7
FOR J=(4,1,5) DO BEGIN  RD(I,J)=ID(I,J)=0.0  $                            7
FOR K=(1,1,2) DO BEGIN                                                    7
CXMULT(-RH(I,K),-IH(I,K),RT(K,J),IT(K,J),R6,I6)  $                        7
RD(I,J)=RD(I,J) + R6  $  ID(I,J)=ID(I,J) + I6  END END END  $             7
```

```
FOR I=(1,1,2) DO BEGIN  FOR J=(1,1,3) DO BEGIN
CXMULT(RD(I,J),ID(I,J),RJJ(J),IJJ(J),RD(I,J),ID(I,J))  $
CXDIV(RD(I,J),ID(I,J),RLAM(J),ILAM(J),RD(I,J),ID(I,J)) END  $
FOR J=(4,1,5) DO BEGIN
CXMULT(RD(I,J),ID(I,J),RHH(J),IHH(J),RD(I,J),ID(I,J))  $
CXDIV(RD(I,J),ID(I,J),RLAM(J),ILAM(J),RD(I,J),ID(I,J)) END END  $
                                                                COMMENT
THIS ROUTINE DIVIDES EACH COLUMN BY ITS LARGEST MAGNITUDE ELEMENT.  $
IF NOT GATE THEN BEGIN
FOR J=(1,1,5) DO BEGIN
RC(J)=RD(1,J)  $  IC(J)=ID(1,J)  $
FOR I=(2,1,5) DO BEGIN
IF AV(RC(J),IC(J)) LSS AV(RD(I,J),ID(I,J)) THEN BEGIN
RC(J)=RD(I,J)  $  IC(J)=ID(I,J)  END END  $
IF AV(RC(J),IC(J)) LSS &&-10 THEN WRITE('D IS SINGULAR/COLUMN J=',J).
END  $
GATE=TRUE END  $
FOR J=(1,1,5) DO FOR I=(1,1,5) DO
CXDIV(RD(I,J),ID(I,J),RC(J),IC(J),RD(I,J),ID(I,J))  $                  10
DETRM(RD,ID,RDET,IDET,5)  $                                            10
FUNCTION=&&+9*AV(RDET,IDET) END FUNCTION  $                            8

PROCEDURE LINEMIN (S)  $
REAL 2 ARRAY S  $
BEGIN
INTEGER  J,COUNT  $
FORMAT OUT(X3,'RK=',R14.8,X3,'IK=',R14.8,X3,'FINAL VALUE=',R14.8,A2.1)$
REAL 2 B,VALB,VALC,ALPHAA,ASTEP  $
REAL 2 ARRAY XB,XC(1..N)  $
FOR J = (1,1,N)  DO
IF ABS(S(J)/Y(J)) GTR B THEN B = ABS(S(J)/Y(J))  $
FOR J = (1,1,N)  DO
XB(J)=Y(J) + &&-6*S(J)/B  $
VALB = FUNCTION (XB)  $
VAL = FUNCTION (Y)  $
WRITE(Y)  $
WRITE('INITIAL FUNCTION VALUE = ', VAL )  $
WRITE('RLAM,ILAM',RLAM,ILAM)  $
IF VAL LSS VALB THEN
ALPHAA = - DELTAB/B ELSE
ALPHAA = DELTAB/B  $
COUNT = 0  $
P0..
COUNT = COUNT + 1  $
FOR J = (1,1,N)  DO  XB(J) = Y(J) + ALPHAA * S(J)  $
VALB = FUNCTION(XB)  $
A1..
IF COUNT EQL 1 THEN  BEGIN
IF VALB LSS VAL THEN  BEGIN

FOR J = (1,1,N)  DO  XC(J) = XB(J)  $
VALC = VALB  $  ASTEP = ALPHAA  $  ALPHAA = 2.0 * ALPHAA  $
GO TO P0  END
ELSE  BEGIN
```

```
ASTEP = 0.5 * ALPHAA  $

FOR J = (1,1,N)  DO  XC(J) = Y(J) + ASTEP * S(J)  $
VALC = FUNCTION (XC)  $
IF VALC LEQ VAL THEN GO TO A2
ELSE  BEGIN
VALB = VALC  $  ALPHAA = 0.5 * ALPHAA  $
FOR J = (1,1,N)  DO  XB(J) = XC(J)  $
COUNT = 1  $
GO TO A1
END  END  END  $
IF VALB LSS VALC THEN  BEGIN
ASTEP = ALPHAA - ASTEP  $  ALPHAA = 1.5 * ALPHAA  $
VAL = VALC  $  VALC = VALB  $
FOR J = (1,1,N)  DO  BEGIN
Y(J) = XC(J)  $  XC(J) = XB(J)  END  $
GO TO PO  END  $
A2..
ALPHAA = 0.5 * (VALB * ASTEP * ASTEP - VALC * ALPHAA * ALPHAA
- VAL * (ASTEP * ASTEP - ALPHAA * ALPHAA))/(VALB * ASTEP
- VALC * ALPHAA - VAL * (ASTEP - ALPHAA))  $
FOR J = (1,1,N)  DO  XB(J) = Y(J) + ALPHAA * S(J)  $
VALB = FUNCTION (XB)  $
IF VALB LEQ VALC THEN  BEGIN
FOR J = (1,1,N)  DO  Y(J) = XB(J)  $
VAL = VALB  $  GO TO A3   END  $
FOR J = (1,1,N)  DO  Y(J) = XC(J)  $
VAL = VALC  $
A3..
WRITE(OUT,Y(2),ISCALE*Y(1),VAL)  $
NEWVAL=VAL  $
END LINEMIN  $

PROCEDURE MINIMIZE (X,N,VAL,FUNCTION)  $
VALUE N $ INTEGER N  $
REAL 2 VAL  $
REAL 2 ARRAY X  $
REAL 2 PROCEDURE FUNCTION  $

BEGIN
INTEGER J,R,K,M  $
REAL 2 DELTA  $
REAL 2 ARRAY X0,X3,ZETA(1..N),S(1..N,1..N),F(0..(N+1))  $
FOR J = (1,1,N)  DO  BEGIN
FOR K = (1,1,N)  DO
S(J,K) = 0.0  $
S(J,J) = 1.0  END  $
F(0) = FUNCTION(X)  $
A1..
FOR J = (1,1,N)  DO X0(J) = X(J)  $  DELTA = 0.0  $
FOR R = (1,1,N)  DO  BEGIN
FOR J = (1,1,N)  DO ZETA(J) = S(J,R)  $
LINEMIN (ZETA)  $
F(R) = VAL  $
```

```
IF F(R-1) - VAL GTR DELTA THEN  BEGIN
DELTA = F(R-1) - VAL  $  M = R  END  .
END  $
IF NEWVAL/OLDVAL GTR 0.999 THEN GO TO A2  $
OLDVAL=NEWVAL  $
F(1) = F(0)  $  F(2) = F(N)  $
FOR J = (1,1,N)  DO  X3(J) = 2.0*X(J) - X0(J)  $
F(3) = FUNCTION(X3)  $
IF F(3) GEQ F(1) OR (F(1) - F(2) - F(2) + F(3))*(F(1) - F(2) - DELTA)**2
GEQ 0.5 * DELTA * (F(1) - F(3))**2  THEN BEGIN F(0)=F(2)$ GO TO A1 END$
FOR J = (M,1,N-1)  DO
FOR K = (1,1,N)  DO
S(K,J)  =  S(K,J+1)  $
FOR K = (1,1,N)  DO
S(K,N)  =  ZETA(K)  =  X(K) - X0(K)  $
LINEMIN (ZETA)  $
F(0) = VAL  $
IF NEWVAL/OLDVAL LSS 0.999 THEN BEGIN OLDVAL=NEWVAL  $  GO TO A1  END  $
A2..
END MINIMIZE  $

                                                          COMMENT
READ AND WRITE THE INPUT DATA.  $
START..
READ(INLIST,NOMO)  $
WRITE(FORM1,INLIST)  $
N =2  $
GATE=FALSE  $
OLDVAL=1.0  $
Y(1)=Y(2)=YSCALE*(A*H/C)  $
                                                          COMMENT
A=OMEGA/OMEGA P,B=OMEGA B/OMEGA P,C=V0/C, D=UP/C,E=UB/C,F=NU P/OMEGA P,
G= NU B/ OMEGA B,H=OMEGA P/C, OMEGA 3 = EB/M.  $
    PH(0) =0$ FOR I=(1,1,250) DO PH(I)=PH(I-1)+(1/I)  $
    FCT(0)=1$ FOR I=(1,1,25) DO FCT(I)=FCT(I-1)*I  $
    PR0(-1)=1$ FOR I=(1,2,17) DO PR0(I)=PR0(I-2)*I*I  $
    PR1(-1)=1$ FOR I=(1,2,17) DO PR1(I)=PR1(I-2)*(4-(I*I))  $
T=2.21067*&&+5  $  A2=A*A  $  B2=B*B  $  D2=D*D  $  E2=E*E  $  H2=H*H  $
C1=A2*H2  $  C2=T*A  $  C3=T/D2  $  C4=T/E2  $  C11=H2/D2  $  C13=H2/E2$
C14=(A*B2*C*H*H2)/T  $  C17=B2*C*H  $  C18=B2*C11  $  C21=B2*C*C*C13  $
C22=(T*A*C*H)/E2  $  C23=B2*C13  $
A2T=A2*T  $  A2H=A2*H  $  AH=A*H  $  A2H2=A2*H2  $  AH2=A*H2  $
MINIMIZE(Y,N,VAL,FUNCTION)  $
GO TO START  $
NOMO..
END$
```

```
LISTING TIME WAS:     5.18 SECOND(S)
  BPA             SYMBOLIC ALGOL      14      14      1                 0  03634164    14    690  (DELETED)
```

REFERENCES

1. Langmuir, I., "Scattering of Electrons in Ionized Gases", Phys. Rev., 26, 585 (1925).

2. Hull, A. W., Phys. Rev., 18, 31 (1921).

3. Heil, A. A., and Heil, O., "A New Method of Generating Short Undamped Electromagnetic Waves", Z. fur Phys., 95, 752 (1935).

4. Varian, R. H., and Varian, S. F., " A High-Frequency Oscillator and Amplifier", J. Appl. Phys., 10, 321, May (1939).

5. Hahn, W. C. and Metcalf, G. F., "Velocity Modulated Tubes", Proc. I.R.E., 27, 106-125, Feb. (1939).

6. Kompfner, R., "Traveling Wave Value", Wireless World, 54, 369-372, November (1946).

7. Pierce, J. R., and Field, L. M., "Traveling-Wave Tubes", Proc. I.R.E., 35, 108-111, February (1947).

8. Jones, E. M. T., "An Internal-Feedback Traveling-Wave-Tube Oscillator", Proc. I.R.E., 40, 478, April (1952).

9. Heffner, H., "Analysis of the Backward-Wave Traveling-Wave Tube", Tech. Rept. No. 48, Electronics Research Laboratory, Stanford University, June (1952).

10. Lord Rayleigh, "On Electrical Vibrations and the Constitution of the Atom", Rhil. Mag., 11, 117-123, January (1906).

11. Pierce, J. R., "Possible Fluctuations in Electron Streams due to Ions", J. Appl. Phys., 19, 231 (1948).

12. Haeff, A. V., "Space-Charge Wave Amplification Effects", Phys. Rev., 74, 1532, November (1948).

13. Nergaard, L. E., "Analysis of a Simple Model of Two-Beam Growing-Wave Tube", R.C.A. Review, 9, 585-601, December (1948).

14. Pierce, J. R. and Herbenstreit, W. B., "A New Type of High-Frequency Amplifier", Bell System Tech. Journal, 28, 33-51, January (1949).

15. Haeff, A. V., "The Electron Wave Tube--A Novel Method of Generation and Amplification of Microwave Energy", Proc. I.R.E., 37, 4-10, January (1949).

16. Crawford, F. W. and Kino, G. S., "Oscillations and Noise in Low-Pressure D.C. Discharges", Proc. I.R.E., 49, 1767-1788, December (1961).

17. Fainberg, Ya. B., "The Interaction of Charged Particle Beams with Plasma", J. Nuclear Energy, Part C, 4, 203 (1962).

18. Briggs, R. J., Electron Stream Interaction with Plasmas, M.I.T. Press, Cambridge, Mass. (1964).

19. Crawford, F. W., "Beam-Plasma Interaction in a Warm Plasma", S.U.I.P.R. Rept. No. 13, Stanford University, May (1965).

20. "Bibliography of Stanford Reports, Papers and Conference Presentations on Plasma Physics", S.U.I.P.R. Rept. No. 52, Stanford University, February (1966).

21. Self, S. A., "Instability and Amplification in Finite Beam-Plasma Systems", Microwave Laboratory Repts., 1292, 1293, Stanford University, January (1965).

22. Bresler, A. D., and Marcuvitz, N., Repts. MRI-R-495-56 (1956), MRI-R-565-57 (1957), Microwave Res. Inst., Polytech. Inst. of Brooklyn.

23. Sancer, M. I., "Analysis of Compressible Plasma Contained in a Waveguide", AFCRL-65-644, Rept. No. 11, Case Inst. of Tech., November (1965).

24. Chen, H. H. C., and Cheng, D. K., "Concerning Lossy, Compressible Magneto-Ionic Media--General Formulation and Equation Decoupling", I.E.E.E. Transactions, AP-14, No. 4, July (1966).

25. Nield, P. W., "Propagation in Waveguides Filled with Warm Plasma", Ph.D. Dissertation, Case Inst. of Tech., June (1967).

26. Pease, M. C., III, Methods of Matrix Algebra, Academic Press, New York (1965).

27. See for example: Collin, R. E., Foundations for Microwave Engineering, McGraw-Hill (1966).

28. Hahn, W. C., "Small Signal Theory of Velocity Modulated Electron Beams", Gen. Elec. Rev., 42, 258-270 (1939).

29. Cannara, A. B., and Crawford, F. W., "Electron Beam Probing Studies of Beam/Plasma Interactions", S.U.I.P.R. Rept. No. 90, Stanford University, August (1966).

30. Kollettis, N. J., "Theoretical and Experimental Study of a Surface Wave on a Temperate Collisional Plasma Column", Rept. No. 14, AFCRL-66-765, November (1966).

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Case Institute of Technology University Circle, Cleveland, Ohio 44106	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

"WAVE INTERACTION IN NONRELATIVISTIC ELECTRON BEAM-PLASMA SYSTEMS"

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim Report

5. AUTHOR(S) (Last name, first name, initial)

Otteni, Gerald A.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July, 1967	149	30

8a. CONTRACT OR GRANT NO. AF19(628)1699	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT AND TASK NO. Work Unit Nos. 5635- 04 01	Scientific Report No. 17
c. DOD ELEMENT 61445014	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d. DOD SUBELEMENT 681305	AFCRL 67-0375

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	A.F. Cambridge Research Labs (CRD) L.G. Hanscom Field Bedford, Massachusetts 01730

13. ABSTRACT

A theoretical model based on the linearization of the fluid equations and Maxwell's equations for wave interaction in a uniform plasma which is interpenetrated by a nonrelativistic electron beam is developed. The effects of electron-neutral and electron-ion collisions and temperatures of both the beam and plasma electrons are included and no quasi-static approximation is made for the electromagnetic field. An external d.c. magnetic field is assumed to act so that a general formulation is developed which is valid in the limit of small d.c. magnetic fields and in the limit as the field becomes very large. Graphs of the computer solutions are given for the propagation constants in a beam-plasma system for the cases of an unbounded system and for the TM wave solutions that may exist in an axisymmetric cylindrical system in which the finite beam interpenetrates an unbounded plasma.

DD FORM 1 JAN 64 1473

Unclassified

Security Classification